

On locally conformal symplectic manifolds of the first kind

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We present some examples of locally conformal symplectic structures of the first kind on compact nilmanifolds which do not admit Vaisman metrics. One of these examples does not admit locally conformal Kähler metrics and all the structures come from left-invariant locally conformal symplectic structures on the corresponding nilpotent Lie groups. Under certain topological restrictions related with the compactness of the canonical foliation, we prove a structure theorem for locally conformal symplectic manifolds of the first kind. In the non compact case, we show that they are the product of a real line with a compact contact manifold and, in the compact case, we obtain that they are mapping tori of compact contact manifolds by strict contactomorphisms. Motivated by the aforementioned examples, we also study left-invariant locally conformal symplectic structures on Lie groups. In particular, we obtain a complete description of these structures (with non-zero Lee 1-form) on connected simply connected nilpotent Lie groups in terms of locally conformal symplectic extensions and symplectic double extensions of symplectic nilpotent Lie groups. In order to obtain this description, we study locally conformal symplectic structures of the first kind on Lie algebras.

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1 Introduction

A locally conformal symplectic structure on a manifold M consists of a pair (Φ, ω) , where Φ and ω are a 2-form and a 1-form, respectively, with Φ non degenerate, ω closed, subject to the equation $d\Phi = \omega \wedge \Phi$. ω is known as the Lee form. This implies that, locally, Φ is conformal to a genuine symplectic form, hence the name. If $\omega = df$, then the global conformal change $\Phi \mapsto e^{-f}\Phi$ endows M with a symplectic structure. We use the name globally conformal symplectic structure in this case. Notice that a manifold endowed with a locally conformal symplectic structure is orientable and almost complex. A locally conformal symplectic manifold is one endowed with a locally conformal symplectic structure.

Locally conformal symplectic structures were introduced by Lee in [39] and then studied extensively by Vaisman (see [72]), Banyaga (see [5, 6, 7]) and many others (see [3, 29, 38, 45]). Vaisman pointed out in [72, Section 1] that locally conformal symplectic structures play an important role in Hamiltonian mechanics, generalizing the usual description of the phase space in terms of symplectic geometry.

In this paper we are mainly concerned with locally conformal symplectic structures of the first kind; this means that there exists an automorphism U of (Φ, ω) with $\omega(U) = 1$; U is then called an anti-Lee vector field. This is equivalent to the existence of a 1-form η , with $d\eta$ of rank $2n - 2$, such that $\Phi = d\eta - \omega \wedge \eta$; here $2n = \dim M$. In order to obtain examples of locally conformal symplectic manifolds, one can take the product of a contact manifold and an interval; more generally, locally

conformal symplectic structures of the first kind exist on the suspension (or mapping torus) of a strict contactomorphism¹ of a contact manifold. We refer to Section 2 for all the details. Banyaga proved a sort of converse to this result: in [5, Theorem 2], he showed that a compact manifold endowed with a locally conformal symplectic structure of the first kind fibres over the circle and that the fibre inherits a contact structure. This result, which shows an interplay between locally conformal symplectic geometry and contact geometry, is a non-metric version of a result of Ornea and Verbitski. In fact, in [59], they proved that compact Vaisman manifolds (see the definition below) are diffeomorphic to mapping tori with Sasakian fiber.

Our first result is the following (see Theorem 4.7):

Theorem A *Let M be a connected manifold endowed with a locally conformal symplectic structure of the first kind (Φ, ω) , where $\Phi = d\eta - \omega \wedge \eta$. Assume that the anti-Lee vector field U is complete and that the foliation $\mathcal{F} = \{\omega = 0\}$ has a compact leaf L . Denote by Ψ the flow of U and write $\eta_L = i^*\eta$, where $i: L \rightarrow M$ is the canonical inclusion. Then we have two possibilities:*

1. *(Φ, ω) is a globally conformal symplectic structure on M and $\Psi: L \times \mathbb{R} \rightarrow M$ is an isomorphism of globally conformal symplectic manifolds;*
2. *there exist a real number $c > 0$ and a strict contactomorphism $\phi: L \rightarrow L$ such that Ψ induces an isomorphism between the locally conformal symplectic manifold of the first kind $L_{(\phi, c)}$ and M . In particular, M is compact.*

In both cases, each leaf of \mathcal{F} is of the form $\Psi_t(L)$ for some $t \in \mathbb{R}$ and $\Psi_t|_L: L \rightarrow \Psi_t(L)$ is a strict contactomorphism.

We refer to the discussion after the proof of Theorem 4.7 (in Section 4.2) for a comparison between Banyaga's result and the previous theorem.

Continuing our parallel between contact and locally conformal symplectic structures of the first kind, a theorem of Martinet (see [47]) asserts that every oriented closed manifold of dimension 3 has a contact form. Hence every orientable closed 3-manifold admits a contact structure. We provide a Martinet-type result for locally conformal symplectic structures of the first kind on 4-manifolds.

In fact, if M is an oriented connected manifold of dimension 4, ω is a closed 1-form on M without singularities and L is a compact leaf of the foliation $\mathcal{F} = \{\omega = 0\}$ then we obtain sufficient conditions for M to admit a locally conformal symplectic structure of the first kind (see Corollary 4.12).

Locally conformal symplectic structures also appear in the context of almost Hermitian geometry. An almost Hermitian structure on a manifold M of dimension $2n$ consists of a Riemannian metric g and a compatible almost complex structure J , that is, an endomorphism $J: TM \rightarrow TM$ with $J^2 = -\text{Id}$, such that $g(JX, JY) = g(X, Y)$ for every $X, Y \in \mathfrak{X}(M)$. If (g, J) is an almost Hermitian structure on M , one can consider the following tensors:

- a 2-form Φ , the Kähler form, defined by $\Phi(X, Y) = g(JX, Y)$, $X, Y \in \mathfrak{X}(M)$;
- a 1-form ω , the Lee form, defined by $\omega(X) = \frac{-1}{n-1} \delta \Phi(JX)$, where δ is the codifferential and $X \in \mathfrak{X}(M)$.

In [28], Gray and Hervella classified almost Hermitian structures (g, J) by studying the covariant derivative of the Kähler form Φ with respect to the Levi-Civita connection ∇ of g . Of particular interest for us are the following classes:

- the class of Kähler structures, for which $\nabla \Phi = 0$. In this case, g is said to be a Kähler metric. A Kähler manifold is a manifold endowed with a Kähler structure. The Lee form is zero on a Kähler manifold. We refer to [34] for an introduction to Kähler geometry.

¹In this paper we will restrict to contact *forms* rather than contact structures. Consequently, our morphisms will be *strict* contactomorphisms, i.e. contactomorphisms preserving the contact form - see Definition 4.3.

- the class of locally conformal Kähler (lcK) structures, for which $d\Phi = \omega \wedge \Phi$; in this case we call g a locally conformal Kähler metric. A locally conformal Kähler manifold is a manifold endowed with a locally conformal Kähler structure. Locally conformal Kähler manifolds were introduced by Vaisman in [67]. A remarkable example of locally conformal Kähler manifold is the Hopf surface. The study of locally conformal Kähler metrics on compact complex surfaces was undertaken in [13]. Homogeneous locally conformal Kähler and Vaisman structures have been studied in [1, 26, 32, 33]. The standard reference for locally conformal Kähler geometry is [22]; see also the recent survey [58].

In the Kähler and locally conformal Kähler case, the almost complex structure J is integrable, hence these are complex manifolds. One can interpret Kähler structure as a “degenerate” case of locally conformal Kähler structures. Indeed, it turns out that the Lee form of a locally conformal Kähler structure is closed; if it is exact, then one can show that there is a Kähler metric in the conformal class of g . In such a case, one says that the structure is globally conformal Kähler. In general, if a manifold admits a genuine locally conformal Kähler structure, i.e. one for which the Lee form is not exact, then it admits an open cover such that the Lee form is exact on each open set, hence the locally conformal Kähler metric is locally conformal to a Kähler metric.

The non-metric version of Kähler manifolds are symplectic manifolds (see [50]). A manifold M^{2n} is symplectic if there exists a 2-form Φ such that $d\Phi = 0$ and Φ^n is a volume form. Clearly, the Kähler form of a Kähler structure is symplectic. It is well known that the existence of a Kähler metric on a compact manifold deeply influences its topology. In fact, suppose M is a compact Kähler manifold of dimension $2n$ and let Φ be the Kähler form; then:

- the odd Betti numbers of M are even;
- the Lefschetz map $H^p(M) \rightarrow H^{2n-p}(M)$, $[\alpha] \mapsto [\alpha \wedge \Phi^{n-p}]$ is an isomorphism for $0 \leq p \leq n$;
- M is formal.

For a long time, the only known examples of symplectic manifolds came from Kähler (or even projective) geometry. In [63], Thurston constructed the first example of a compact, symplectic 4-manifold which violates each of the three conditions given above. Since then, the problem of constructing compact symplectic manifolds without Kähler structures has inspired beautiful Mathematics (see, for instance, [23, 27, 49, 57]).

Following this train of thought, the Kähler form and the Lee form of a locally conformal Kähler structure define a locally conformal symplectic structure. In contrast to the Kähler case, however, not much is known about the topology of compact locally conformal Kähler manifolds. In [22, Conjecture 2.1] it was conjectured that a compact locally conformal Kähler manifold which satisfies the topological restrictions of a Kähler manifold admits a (global) Kähler metric. A stronger conjecture (see [70]) is that a compact locally conformal Kähler manifold which is not globally conformal Kähler must have at least one odd degree Betti number which is odd. It was shown in [10, 37, 71] that a compact Vaisman manifold, i.e. a locally conformal Kähler manifold with non-zero parallel Lee form (see [68]), has odd b_1 . Hence the stronger conjecture holds for Vaisman structures. However, it does not hold in general: in [56], Oeljeklaus and Toma constructed a locally conformal Kähler manifold of complex dimension 3 with $b_1 = b_5 = 2$, $b_0 = b_2 = b_4 = b_6 = 1$ and $b_3 = 0$.

In [58], the authors proposed the following problem:

Is there a compact manifold with locally conformal symplectic structures but no locally conformal Kähler metric?

A first answer to this question was given by Bande and Kotschik: in [4] they described a locally conformal symplectic structure on a 4-manifold of the form $M \times S^1$, which does not carry any complex structure, hence no locally conformal Kähler metric.

The second goal of this paper is to give a different answer to the question above. In fact, we prove

the following result (see Corollary 3.6):

Theorem B *There exists a compact, 4-dimensional nilmanifold, not diffeomorphic to the product of a compact 3-manifold and a circle, which has a locally conformal symplectic structure but no locally conformal Kähler metric.*

This result is contained in the preprint [11], of which this paper represents a substantial expansion. The example of Thurston that we mentioned above is also a nilmanifold. This makes the parallel between the symplectic and the locally conformal symplectic case particularly transparent. The product of the Heisenberg manifold and the real line admits a compact quotient, which turns out to be a nilmanifold which admits a Vaisman structure. It was conjectured by Ugarte in [66] that this is basically the only possibility, i.e. that a compact nilmanifold endowed with a locally conformal Kähler structure with non-zero Lee form is a compact quotient of the Heisenberg group multiplied by \mathbb{R} . This conjecture was proved by Sawai in [62], assuming that the locally conformal Kähler structure has a left-invariant complex structure, but remains open in general. In Section 3 we provide different examples of compact nilmanifolds with locally conformal symplectic structures.

On the other hand, in last years, special attention has been devoted to the study of symplectic Lie algebras (see [9, 18, 43, 51, 60]). In particular, in [51] (see also [18]), the authors introduce the notion of a symplectic double extension of a symplectic Lie algebra. In fact, if \mathfrak{s}_1 is a symplectic Lie algebra of dimension $2n - 2$ then, in the presence of a derivation on \mathfrak{s}_1 and an element of \mathfrak{s}_1 which satisfy certain conditions, one may produce a new symplectic Lie algebra of dimension $2n$. In addition, in [51] (see also [18]), the authors prove a very interesting result: *every symplectic nilpotent Lie algebra of dimension $2n$ may be obtained as a sequence of $(n - 1)$ symplectic double extensions from the abelian Lie algebra \mathbb{R}^2 .*

We remark that a symplectic structure on a Lie algebra \mathfrak{s} induces a left-invariant symplectic structure on a Lie group with Lie algebra \mathfrak{s} and, conversely, a left-invariant symplectic structure on a Lie group induces a symplectic structure on its Lie algebra. Furthermore, symplectic structures may be considered as locally conformal symplectic structures with zero Lee 1-form. In addition, all the locally conformal symplectic structures on the examples of compact nilmanifolds in Section 3 come from left-invariant locally conformal symplectic structures (with non-zero Lee 1-form) on connected simply connected Lie groups.

Therefore, the following problem we tackle in this paper is the study of left-invariant locally conformal symplectic structures on Lie groups with non-zero Lee 1-form and, more precisely, left-invariant locally conformal symplectic structures of the first kind. In this direction, our first result relates locally conformal symplectic Lie groups of the first kind with contact Lie groups (that is, Lie groups endowed with left-invariant contact structures). In fact, we prove that (see Section 6.1):

Theorem C *The extension of a contact Lie group H by a contact representation of the abelian Lie group \mathbb{R} on H is a locally conformal symplectic Lie group of the first kind. Conversely, every connected simply connected locally conformal symplectic group of the first kind is an extension of a connected simply connected contact Lie group H by a contact representation of \mathbb{R} on H .*

We also introduce the definition of a locally conformal symplectic extension of a symplectic Lie group S by a symplectic 2-cocycle and a symplectic representation of \mathbb{R} on S and, then, we prove the following result (see Section 6.2):

Theorem D *The locally conformal symplectic extension of a symplectic Lie group S of dimension $2n$ is a locally conformal symplectic Lie group of the first kind with bi-invariant Lee vector field and dimension $2n + 2$. Conversely, every connected simply connected locally conformal symplectic Lie group of the first kind with bi-invariant Lee vector field is the locally conformal symplectic extension of a connected simply connected symplectic Lie group.*

The last part of the paper is devoted to the study of locally conformal symplectic nilpotent Lie groups with non-zero Lee 1-form. We completely describe the nature of these Lie groups in terms

of locally conformal symplectic extensions and double symplectic extensions. In fact, we prove the following result (see Theorem 6.14):

Theorem E *Every connected simply connected locally conformal symplectic nilpotent Lie group of dimension $2n + 2$ with non-zero Lee 1-form is the locally conformal symplectic nilpotent extension of a connected simply connected symplectic nilpotent Lie group S and, in turn, S may be obtained as a sequence of $(n - 1)$ double symplectic nilpotent extensions from the abelian Lie group \mathbb{R}^2 .*

We show that all the compact locally conformal symplectic nilmanifolds in Section 3 may be described using the previous result. On the other hand, in order to obtain the above results on locally conformal symplectic Lie groups, we discuss Lie algebras endowed with locally conformal symplectic structures.

This paper is organized as follows:

- in Section 2 we recall the main definitions and known results about locally conformal symplectic geometry, Lie algebra cohomology, multiplicative vector fields, central extensions of Lie algebras and groups and compact nilmanifolds;
- in Section 3 we obtain some examples of compact locally conformal symplectic nilmanifolds of the first kind (symplectic or not) which do not admit Vaisman metrics; in particular, we present an example of a compact, 4-dimensional nilmanifold, not diffeomorphic to the product of a compact 3-manifold and a circle, which has a locally conformal symplectic structure but no locally conformal Kähler metric;
- in Section 4 we provide some general results about foliations of codimension 1, we describe the global structure of a connected locally conformal symplectic manifold of the first kind with a compact leaf in its canonical foliation (see Theorem 4.7) and we deduce some consequences (see Corollary 4.12);
- in Section 5 we study locally conformal symplectic structures on Lie algebras, with a particular emphasis on the nilpotent case.
- in Section 6 we consider left-invariant locally conformal symplectic structures on Lie groups emphasizing again the nilpotent case. We show how to recover the examples of Section 3 from the general description.

2 Preliminaries

In this section we review the basics in locally conformal symplectic geometry as well as some definitions, constructions and results on Lie algebra cohomology, multiplicative vector fields, central extensions of Lie algebras and groups and compact nilmanifolds.

2.1 Locally conformal symplectic structures

Definition 2.1. A *locally conformal symplectic (lcs)* structure on a smooth manifold M^{2n} ($n \geq 2$) consists of a 2-form $\Phi \in \Omega^2(M)$ and a closed 1-form $\omega \in \Omega^1(M)$, called the Lee form, such that Φ^n is a volume form on M and

$$d\Phi = \omega \wedge \Phi. \quad (1)$$

The lcs structure is *globally conformal symplectic (gcs)* if ω is exact.

Notice in particular that if $H^1(M; \mathbb{R}) = 0$, every lcs structure on M is gcs. We require $n \geq 2$, since in dimension 2 a 2-form is automatically closed, hence lcs geometry in dimension 2 is nothing but symplectic geometry.

Here is an equivalent definition: a manifold M^{2n} has a lcs structure if there exist a 2-form $\Phi \in \Omega^2(M)$ with $\Phi^n \neq 0$, an open cover $M = \cup_\alpha U_\alpha$ and smooth functions $\sigma_\alpha: U_\alpha \rightarrow \mathbb{R}$ such that $\Phi_\alpha = e^{\sigma_\alpha} \Phi|_{U_\alpha}$ satisfies $d\Phi_\alpha = 0$. The lcs structure is globally conformal symplectic if the domain of σ_α can be chosen to be all of M . In this case, the global conformal change $\Phi \mapsto \Phi' = e^\sigma \Phi$ produces a *closed* and non-degenerate 2-form Φ' , hence (M, Φ') is a symplectic manifold.

Since Φ is a non-degenerate 2-form, it provides an isomorphism $\mathfrak{X}(M) \rightarrow \Omega^1(M)$ given by

$$X \mapsto \iota_X \Phi.$$

We define $V \in \mathfrak{X}(M)$ by the condition $\iota_V \Phi = \omega$. V is the *Lee vector field* of the lcs structure. Clearly $\omega(V) = 0$, since Φ is skew-symmetric.

As usual, a good approach to the study of a geometric structure is to consider its automorphisms. If (Φ, ω) is a lcs structure on M , an *infinitesimal automorphism* of (Φ, ω) is a vector field $X \in \mathfrak{X}(M)$ such that $\mathcal{L}_X \Phi = 0$. Since $n \geq 2$, $\mathcal{L}_X \omega = 0$. Infinitesimal automorphisms of (Φ, ω) form a Lie subalgebra $\mathfrak{X}_\Phi(M)$ of $\mathfrak{X}(M)$. Moreover, if $X \in \mathfrak{X}_\Phi(M)$ then $\omega(X)$ is a constant function whenever M is connected. Hence the *Lee morphism* $\ell: \mathfrak{X}_\Phi(M) \rightarrow \mathbb{R}$, $\ell(X) = \omega(X)$, is well defined. Viewing \mathbb{R} as an abelian Lie algebra, ℓ becomes a Lie algebra morphism (see [72] or [7, Proposition 2]). Notice that the Lee vector field V is in $\mathfrak{X}_\Phi(M)$ and that $\ell(V) = 0$. Since the image of ℓ has dimension at most 1, the Lee morphism is either surjective or identically zero. The lcs structure (Φ, ω) is said to be *of the first kind* if the Lee morphism is surjective, *of the second kind* otherwise (see [72]).

In this paper, we shall restrict ourselves to lcs structures of the first kind. Also, in general, when we say *lcs structure* we explicitly exclude the possibility that the structure is actually globally conformal symplectic.

Let (Φ, ω) be a lcs structure of the first kind on M . Let $U \in \mathfrak{X}_\Phi(M)$ be a vector field such that $\omega(U) = 1$. Define $\eta \in \Omega^1(M)$ by the equation $\eta = -\iota_U \Phi$. If U and η are as above and V is the Lee vector field then

$$\eta(U) = 0, \quad \eta(V) = 1, \quad \mathcal{L}_U \eta = \mathcal{L}_U \omega = \mathcal{L}_V \eta = \mathcal{L}_V \omega = 0.$$

In particular, $\iota_U d\eta = \iota_V d\eta = 0$.

The next proposition gives an alternative characterization of lcs structures of the first kind in terms of the existence of a certain 1-form. For a proof, see [72, Proposition 2.2].

Proposition 2.2. *Let M^{2n} be manifold endowed with a lcs structure of the first kind (Φ, ω) . Then there exists a 1-form $\eta \in \Omega^1(M)$ such that*

$$\Phi = d\eta - \omega \wedge \eta, \tag{2}$$

$d\eta$ has rank $2n-2$ and $\omega \wedge \eta \wedge (d\eta)^{n-1}$ is a volume form. Conversely, let M^{2n} be a manifold endowed with two nowhere zero 1-forms ω and η , with $d\omega = 0$, $\text{rank}(d\eta) < 2n$ and such that $\omega \wedge \eta \wedge (d\eta)^{n-1}$ is a volume form. Then M admits a lcs structure of the first kind.

The conditions given in Proposition 2.2 imply that there exist two vector fields U and V on M which are characterized by the following conditions

$$\begin{aligned} \omega(U) &= 1, & \eta(U) &= 0, & \iota_U d\eta &= 0, \\ \omega(V) &= 0, & \eta(V) &= 1, & \iota_V d\eta &= 0. \end{aligned}$$

Here V is the Lee vector field, U is an *anti-Lee vector field* and η is an *anti-Lee 1-form*.

Note that different anti-Lee vector fields (and, thus, different anti-Lee 1-forms) may exist for a lcs structure of the first kind. However, from now on, when we refer to a lcs structure of the first kind, we will assume that the anti-Lee vector field (and, therefore, the anti-Lee 1-form) is fixed.

Consequently, in what follows, a lcs structure of the first kind will be a couple of 1-forms which satisfy the conditions in Proposition 2.2.

Lcs structures can be told apart according to another criterion, which we now review briefly. Let M be a smooth manifold and let $\omega \in \Omega^1(M)$ be a closed 1-form. In [29], a twisted de Rham differential d_ω was defined: given $\alpha \in \Omega^p(M)$, one sets

$$d_\omega(\alpha) = d\alpha - \omega \wedge \alpha.$$

One sees that $(d_\omega)^2 = 0$, hence the cohomology $H_\omega^\bullet(M)$ of the complex $(\Omega^\bullet(M), d_\omega)$ is defined. $H_\omega^\bullet(M)$ is called the *Lichnerowicz* or *Morse-Novikov* cohomology of M relative to ω . Notice that d_ω is not a derivation with respect to the algebra structure given by the wedge product on $\Omega^\bullet(M)$, i.e. it does not satisfy the Leibniz rule. If ω is an exact 1-form, then $(\Omega^\bullet(M), d_\omega)$ is homotopic to the standard de Rham complex $(\Omega^\bullet(M), d)$, hence $H_\omega^\bullet(M) \cong H^\bullet(M; \mathbb{R})$. But this is not the case if ω is not exact; for instance, when M is connected, oriented and n -dimensional, $H_\omega^n(M) = 0$, see [29, Page 429]. However, as pointed out in [3], the Euler characteristic of the Lichnerowicz cohomology equals the usual Euler characteristic. Also, $H_\omega^\bullet(M)$ is isomorphic to the cohomology of M with values in the sheaf \mathcal{F}_ω of local functions f on M such that $d_\omega f = 0$ (see [69, Proposition 3.1]).

Let (Φ, ω) be a lcs structure on a manifold M . Equation (1) above tells us that Φ is a 2-cocycle in $(\Omega^\bullet(M), d_\omega)$, hence it defines a cohomology class $[\Phi]_\omega \in H_\omega^2(M)$. The lcs structure (Φ, ω) is called *exact* if $[\Phi]_\omega = 0 \in H_\omega^2(M)$, *non exact* otherwise.

Equation (2) above shows that a lcs structure of the first kind is automatically exact. The converse, however, need not be true. Indeed, by [7, Proposition 3], out of an exact lcs structure (Φ, ω) we get a vector field X which satisfies $\mathcal{L}_X \Phi = (\omega(X) - 1)\Phi$ and we can not assure that $\omega(X) = 1$. See also Example 5.4 below.

According to [5, Corollary 1], the study of non exact lcs structures and of their automorphisms on a manifold M is strictly related to the study of symplectic structures on the minimum cover $\tilde{M} \rightarrow M$ on which the Lee form becomes exact and of their corresponding automorphisms. Examples of non exact lcs structures on the 4-dimensional solvmanifold constructed in [2] are given in [5].

We describe now two methods to construct manifolds endowed with lcs structures of the first kind from contact manifolds. These examples play a very important role in this paper.

Example 2.3. Let (L, θ) be a contact manifold of dimension $2n - 1$; hence $\theta \wedge (d\theta)^{n-1}$ is a volume form and the *Reeb field* $R \in \mathfrak{X}(M)$ is uniquely determined by the conditions $\iota_R d\theta = 0$ and $\iota_R \theta = 1$. Let $I \subset \mathbb{R}$ be an open interval (we do not exclude the case $I = \mathbb{R}$). Then $M = L \times I$ admits a gcs structure, which we describe using Proposition 2.2. Let π_i denote the projection from M to the i -th factor, $i = 1, 2$. We have two 1-forms ω_θ and η_θ on M ,

$$\omega_\theta = \pi_2^* dt \quad \text{and} \quad \eta_\theta = \pi_1^* \theta,$$

where t is the standard coordinate on \mathbb{R} . The anti-Lee vector field is $U_\theta = \left(0, \frac{\partial}{\partial t}\right)$ while the Lee vector field is $V_\theta = (R, 0)$. Clearly, M is globally conformal symplectic to the standard symplectization of the contact manifold L .

In the compact case, one can start with a compact contact manifold (L, θ) and consider the product $M := L \times S^1$. The two 1-forms

$$\omega_\theta = \pi_2^* \tau \quad \text{and} \quad \eta_\theta = \pi_1^* \theta,$$

where τ is the angular form on S^1 , give a lcs structure of the first kind on M .

Example 2.4. More generally, let (L, θ) be a contact manifold and let $\phi: L \rightarrow L$ be a strict contactomorphism: this means that ϕ is a diffeomorphism and $\phi^* \theta = \theta$. Let c be a positive real

number and denote by M the *suspension* of L by ϕ and $c > 0$, i.e.

$$M = L \times_{(\phi, c)} \mathbb{R} = \frac{L \times \mathbb{R}}{\sim_{(\phi, c)}},$$

where $(x, t) \sim_{(\phi, c)} (x', t')$ if and only if $x = \phi^k(x')$ and $t' = t + ck$, $k \in \mathbb{Z}$. M is also known as the *mapping torus* $L_{(\phi, c)}$ of L by c and ϕ ; it can alternatively be described as the quotient space

$$M = \frac{L \times [0, c]}{(\phi(x), 0) \sim (x, c)}.$$

The standard gcs structure $(\omega_\theta, \eta_\theta)$ on $L \times \mathbb{R}$ induces a lcs structure of the first kind on M , which we denote by $(\omega_{(\theta, \phi, c)}, \eta_{(\theta, \phi, c)})$. We also have a canonical projection $\pi: M \rightarrow S^1 = \mathbb{R}/c\mathbb{Z}$ satisfying $\omega_{(\theta, \phi, c)} = \pi^*(\tau)$, where τ is the angular form on S^1 . Furthermore, if L is compact, so is M .

2.2 Lie algebra cohomology

Let \mathfrak{g} be a real Lie algebra of dimension n and let W be a finite dimensional \mathfrak{g} -module. For $1 \leq p \leq n$ we consider the space

$$C^p(\mathfrak{g}; W) = \{f: \Lambda^p \mathfrak{g} \rightarrow W\};$$

we also set $C^0(\mathfrak{g}; W) = W$. For $0 \leq p \leq n$ we define a map $\delta: C^p(\mathfrak{g}; W) \rightarrow C^{p+1}(\mathfrak{g}; W)$ by

$$\begin{aligned} (\delta f)(x_1, \dots, x_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} x_i \cdot f(x_1, \dots, \hat{x}_i, \dots, x_{p+1}) \\ &\quad + \sum_{i < j} (-1)^{i+j} f([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{p+1}), \end{aligned}$$

where \cdot denotes the \mathfrak{g} -module structure on W and \hat{x}_i means that x_i is omitted. We set

$$Z^p(\mathfrak{g}; W) = \ker(\delta: C^p(\mathfrak{g}; W) \rightarrow C^{p+1}(\mathfrak{g}; W)) \quad \text{and} \quad B^p(\mathfrak{g}; W) = \text{im}(\delta: C^{p-1}(\mathfrak{g}; W) \rightarrow C^p(\mathfrak{g}; W)).$$

Elements in $Z^p(\mathfrak{g}; W)$ (resp. $B^p(\mathfrak{g}; W)$) are called *p-cocycles* (resp. *p-coboundaries*). One has that $\delta^2 = 0$, hence the degree p cohomology of the complex $(C^*(\mathfrak{g}; W), d)$ can be defined as

$$H^p(\mathfrak{g}; W) = \frac{Z^p(\mathfrak{g}; W)}{B^p(\mathfrak{g}; W)}.$$

Example 2.5. When $W = \mathbb{R}$, the trivial \mathfrak{g} -module, then $(C^*(\mathfrak{g}; W), \delta)$ is isomorphic to the Chevalley-Eilenberg complex $(\Lambda^\bullet \mathfrak{g}^*, d)$, where $\mathfrak{g}^* = \text{hom}(\mathfrak{g}, \mathbb{R})$. In this case, $b_p(\mathfrak{g}) := \dim H^p(\mathfrak{g}; \mathbb{R})$ are the *Betti numbers* of \mathfrak{g} .

Example 2.6. Given a Lie algebra \mathfrak{g} , suppose that there exists a non-zero $\omega \in \mathfrak{g}^*$ with $d\omega = 0$. Consider the 1-dimensional non-trivial \mathfrak{g} -module W_ω with \mathfrak{g} -representation given by

$$X \cdot w = -\omega(X)w, \tag{3}$$

where $w \in W_\omega$. Since $d\omega = 0$, (3) is indeed a Lie algebra representation. In this peculiar situation, a computation shows that $\delta = d_\omega$, where $d_\omega(\alpha) = d\omega - \omega \wedge \alpha$. Thus the Lie algebra cohomology of W_ω coincides with the so-called *Lichnerowicz* or *Morse-Novikov* cohomology of \mathfrak{g} (see [52])

Recall that, for a nilpotent Lie algebra \mathfrak{g} , $b_1(\mathfrak{g}) \geq 2$, hence we can always find a non-zero element $\omega \in \mathfrak{g}^*$ with $d\omega = 0$. In this case, W_ω is a non-trivial 1-dimensional \mathfrak{g} -module. We need the following result of Dixmier:

Theorem 2.7 ([21, Théorème 1]). *Let \mathfrak{g} be a nilpotent n -dimensional Lie algebra and let W be a \mathfrak{g} -module such that every \mathfrak{g} -module contained in W is non-trivial. Then $H^p(\mathfrak{g}; W) = 0$ for $0 \leq p \leq n$.*

Corollary 2.8. *Suppose \mathfrak{g} is an n -dimensional nilpotent Lie algebra and pick a non-zero $\omega \in \mathfrak{g}^*$ with $d\omega = 0$. Then $H^p(\mathfrak{g}; W_\omega) = 0$ for $0 \leq p \leq n$.*

2.3 Multiplicative vector fields on Lie groups

In this section we review some definitions and constructions on multiplicative vector fields in a Lie group and semidirect product of Lie groups and algebras (for more information see, for instance, [48]).

Let H be a Lie group with Lie algebra \mathfrak{h} and let $\phi: \mathbb{R} \rightarrow \text{Aut}(H)$ be a representation of the abelian Lie group \mathbb{R} on H . ϕ is the flow of a multiplicative vector field $\mathcal{M}: H \rightarrow TH$, that is, \mathcal{M} satisfies the condition

$$\mathcal{M}(hh') = \mathcal{M}(h) \cdot \mathcal{M}(h'), \quad \text{for } h, h' \in H$$

where \cdot denotes the multiplication in the Lie group TH . In other words,

$$\mathcal{M}(hh') = (T_h r_{h'}) (\mathcal{M}(h)) + (T_{h'} \ell_h) (\mathcal{M}(h')),$$

$r_{h'}: H \rightarrow H$ and $\ell_h: H \rightarrow H$ being right translation by h' and left translation by h , respectively.

If e is the identity element in H , $\mathcal{M}(e) = 0$ and, in addition, \mathcal{M} induces a derivation $D: \mathfrak{h} \rightarrow \mathfrak{h}$ in the Lie algebra \mathfrak{h} which is defined by

$$\overleftarrow{D}\overleftarrow{X} = [\overleftarrow{X}, \mathcal{M}], \quad (4)$$

for $X \in \mathfrak{h}$, where $[\cdot, \cdot]$ is the standard Lie bracket of vector fields in H . Here, we use the following notation: if K is a Lie group with Lie algebra \mathfrak{k} and $X \in \mathfrak{k}$ then \overleftarrow{X} is the left-invariant vector field on K whose value at e is X . Recall that a derivation of a Lie algebra \mathfrak{h} is a linear map $D: \mathfrak{h} \rightarrow \mathfrak{h}$ such that

$$D([X, Y]) = [D(X), Y] + [X, D(Y)] \quad \forall X, Y \in \mathfrak{h}.$$

The derivations of a Lie algebra \mathfrak{h} form a Lie algebra, denoted $\text{Der}(\mathfrak{h})$. A derivation is a 2-cocycle in the complex $C^*(\mathfrak{h}, \mathfrak{h})$, where the \mathfrak{h} -module structure of \mathfrak{h} is given by the adjoint representation. A derivation D is *inner* if $D = \text{ad}_X$ for some $X \in \mathfrak{h}$.

Using the representation ϕ , we may consider the semidirect product Lie group $G = H \rtimes_{\phi} \mathbb{R}$, with multiplication defined by

$$(h, t)(h', t') = (h\phi_t(h'), t + t'), \quad \text{for } (h, t), (h', t') \in G.$$

The Lie algebra \mathfrak{g} of G is the semidirect product $\mathfrak{g} = \mathfrak{h} \rtimes_D \mathbb{R}$. This is simply $\mathfrak{h} \oplus \mathbb{R}$ with bracket

$$[(X, a), (Y, b)] = (aD(Y) - bD(X) + [X, Y]_{\mathfrak{g}}, 0), \quad \text{for } (X, a), (Y, b) \in \mathfrak{h} \oplus \mathbb{R}. \quad (5)$$

Conversely, let $D: \mathfrak{h} \rightarrow \mathfrak{h}$ be a derivation of \mathfrak{h} and let H be the connected simply connected Lie group with Lie algebra \mathfrak{h} . Then, there exists a unique multiplicative vector field \mathcal{M} on H whose flow induces a representation

$$\phi: \mathbb{R} \rightarrow \text{Aut}(H)$$

of the abelian Lie group \mathbb{R} on H and

$$\overleftarrow{D}\overleftarrow{X} = [\overleftarrow{X}, \mathcal{M}],$$

for $X \in \mathfrak{h}$. Hence we can consider the Lie group $G = H \rtimes_{\phi} \mathbb{R}$ whose Lie algebra \mathfrak{g} is $\mathfrak{h} \rtimes_D \mathbb{R}$.

The semidirect product of a group H and \mathbb{R} with an automorphism $\phi: \mathbb{R} \rightarrow \text{Aut}(H)$ sits in a short exact sequence $1 \rightarrow H \rightarrow H \rtimes_{\phi} \mathbb{R} \rightarrow \mathbb{R} \rightarrow 1$ and H is a normal subgroup of $H \rtimes_{\phi} \mathbb{R}$.

2.4 Central extensions of Lie algebras and groups by \mathbb{R}

In this section we review some constructions on central extensions of Lie algebras (resp. groups) by the abelian Lie algebra (resp. group) \mathbb{R} (see, for instance, [46, 65]).

Let S be a Lie group with Lie algebra \mathfrak{s} . A 2-cocycle on S with values in \mathbb{R} is a smooth map $\varphi: S \times S \rightarrow \mathbb{R}$ such that

$$\varphi(s, s') - \varphi(s, s's'') + \varphi(ss', s'') - \varphi(s', s'') = 0, \quad \text{for } s, s', s'' \in S.$$

If φ is a 2-cocycle on S with values in \mathbb{R} , we can consider the central extension of S by φ , denoted $\mathbb{R} \odot_{\varphi} S$. As a set, this is $\mathbb{R} \times S$, with Lie group structure given

$$(u, s)(u', s') = (u + u' + \varphi(s, s'), ss') \quad \text{for } (u, s), (u', s') \in \mathbb{R} \times S.$$

On the other hand, the 2-cocycle φ induces a 2-cocycle σ on \mathfrak{s} (compare with Section 2.2), defined by

$$\sigma(X, Y) = \left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} (\varphi(\exp(tX), \exp(sY)) - \varphi(\exp(sY), \exp(tX))), \quad (6)$$

where $\exp: \mathfrak{s} \rightarrow S$ is the exponential map associated with the Lie group S .

Moreover, the Lie algebra of the central extension $\mathbb{R} \odot_{\varphi} S$ is the central extension $\mathbb{R} \odot_{\sigma} \mathfrak{s}$ of \mathfrak{s} by σ ; this is just $\mathbb{R} \oplus \mathfrak{s}$ with bracket

$$[(u, X), (u', X')] = (\sigma(X, X'), [X, X']), \quad \text{for } (u, X), (u', X') \in \mathbb{R} \oplus \mathfrak{s}. \quad (7)$$

Conversely, let S be a connected simply connected Lie group with Lie algebra \mathfrak{s} and let $\sigma \in \Lambda^2 \mathfrak{s}^*$ be a 2-cocycle. Then, one may find a 2-cocycle $\varphi: S \times S \rightarrow \mathbb{R}$ on S with values in \mathbb{R} such that φ and σ are related by (6). In addition, the central extension $\mathbb{R} \odot_{\varphi} S$ of S by φ is a Lie group with Lie algebra the central extension $\mathbb{R} \odot_{\sigma} \mathfrak{s}$ of \mathfrak{s} by σ (for more details, see [65]). A central extension of a group S by \mathbb{R} , given by a 2-cocycle $\varphi: S \times S \rightarrow \mathbb{R}$ sits in a short exact sequence $1 \rightarrow \mathbb{R} \rightarrow \mathbb{R} \odot_{\varphi} S \rightarrow S \rightarrow 1$ and \mathbb{R} is a normal subgroup of $\mathbb{R} \odot_{\varphi} S$.

2.5 Compact nilmanifolds

Let G be a connected, simply connected nilpotent Lie group and let \mathfrak{g} be its Lie algebra. It is interesting to know when G contains a discrete, co-compact subgroup (i.e. a lattice).

Theorem 2.9 (Mal'tsev, [44]). *G contains a lattice if and only if there exists a basis of \mathfrak{g} such that the structure constants of \mathfrak{g} with respect to this basis are rational numbers.*

Assume that G contains a lattice Γ and let $N = \Gamma \backslash G$ be the corresponding nilmanifold. One identifies elements of $\Lambda^{\bullet} \mathfrak{g}^*$ with left-invariant forms on G , which descend to N .

Theorem 2.10 (Nomizu, [55]). *The natural inclusion $(\Lambda^{\bullet} \mathfrak{g}^*, d) \hookrightarrow (\Omega^{\bullet}(N), d)$ induces an isomorphism on cohomology.*

2.6 Notation for Lie algebras

We will adopt the following notation for Lie algebras, best explained by an example. $\mathfrak{g} = (0, 0, 0, 12)$ means that \mathfrak{g}^* has a basis $\{e^1, \dots, e^4\}$ such that $de^1 = de^2 = de^3 = 0$ and $de^4 = e^{12}$, where d is the Chevalley-Eilenberg differential and $e^{12} := e^1 \wedge e^2$.

3 Some examples of compact lcs manifolds of the first kind

In this section we use nilmanifolds to construct examples of lcs manifolds of the first kind. In dimension 4, we construct a compact nilmanifold endowed with a lcs structure of the first kind which does not carry any lcK metric and, furthermore, is not the product of a 3-manifold and a circle, thus proving Theorem 3.6. In higher dimension we construct compact nilmanifolds with lcs structures of the first kind, symplectic or not, which do not carry any lcK metric (with left-invariant complex structure) and any Vaisman metric. We summarize our examples in Table 1 (the superscript [†] means *left-invariant* complex or lcK structure).

Table 1: Summary of the examples

	dimension	symplectic	complex [†]	complex	lcs	lcK [†]	lcK	Vaisman
Theorem 3.4	4	✓	×	×	✓	×	×	×
Theorem 3.9	6	✓	✓	✓	✓	×	?	×
Theorem 3.13	6	×	×	?	✓	×	?	×
Theorem 3.16	$2n, n \geq 4$	×	✓	✓	✓	×	?	×

3.1 Dimension 4

In this section we present an example of a 4-dimensional compact nilmanifold endowed with a lcs structure of the first kind, without complex structures, hence no lcK structures. We use the general construction in Example 2.4, i.e. we consider the mapping torus of a compact contact manifold by a strict contactomorphism. The contact manifold is a compact quotient of the Heisenberg group of dimension 3. This is the connected simply connected nilpotent Lie group

$$H = \left\{ \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

We denote a point of H by (x, y, z) . In these global coordinates, a basis of left-invariant 1-forms is given by

$$\alpha = dx, \quad \beta = dy \quad \text{and} \quad \theta = dz - ydx.$$

It is clear that θ is a contact 1-form on H . We look for a strict contactomorphism ϕ of (H, θ) . The condition

$$d(z \circ \phi) - dz = (y \circ \phi)d(x \circ \phi) - ydx$$

must hold and if we assume that

$$z \circ \phi = z + f(y), \quad y \circ \phi = y, \quad x \circ \phi = x + g(y),$$

it follows that

$$\frac{df}{dy} = y \frac{dg}{dy}.$$

In particular, if $t \in \mathbb{R}$ we have that

$$\phi_t: H \rightarrow H, \quad \phi_t(x, y, z) = (x + ty, y, z + ty^2/2)$$

is a strict contactomorphism. In fact, it is easy to prove that

Lemma 3.1. *In the above situation, $\{\phi_t\}_{t \in \mathbb{R}}$ is a 1-parameter group of strict contactomorphisms for (H, θ) and each ϕ_t is a Lie group isomorphism.*

Now, denote by ϕ the strict contactomorphism ϕ_1 . We want to find a lattice $\Gamma \subset H$ such that $\phi(\Gamma) = \Gamma$. It is sufficient to take

$$\Gamma = \{(m, 2n, 2p) \in H \mid m, n, p \in \mathbb{Z}\}.$$

Indeed, we have that $\phi(\Gamma) = \Gamma$, hence

$$\phi_r(\Gamma) = \Gamma, \quad \text{for every } r \in \mathbb{Z}. \quad (8)$$

This implies that ϕ induces a diffeomorphism $\bar{\phi}: L \rightarrow L$, where L is the compact nilmanifold $L = \Gamma \backslash H$.

On the other hand, θ induces a contact 1-form $\bar{\theta}$ on L and it is clear that $\bar{\phi}: L \rightarrow L$ is a strict contactomorphism of the contact manifold $(L, \bar{\theta})$. Thus, the mapping torus $L_{(\bar{\phi}, 1)}$ of L by the couple $(\bar{\phi}, 1)$

$$L_{(\bar{\phi}, 1)} = \frac{L \times \mathbb{R}}{\sim_{(\bar{\phi}, 1)}}$$

is a 4-dimensional compact lcs manifold of the first kind.

Next, we present a description of $L_{(\bar{\phi}, 1)}$ as a compact nilmanifold. In fact, using Lemma 3.1, we deduce that

$$\phi: \mathbb{R} \rightarrow \text{Aut}(H), \quad t \mapsto \phi_t$$

is a representation of the abelian Lie group \mathbb{R} on H . Therefore, we can consider the semidirect product $G = H \rtimes_{\phi} \mathbb{R}$ with multiplication given by

$$((x, y, z), t) \cdot ((x', y', z'), t') = ((x + x' + ty', y + y', z + z' + t(y')^2/2 + yx' + ty y'), t + t').$$

A basis of left-invariant vector fields on G is $\{U, V, A, B\}$, with

$$U = \frac{\partial}{\partial t}, \quad V = \frac{\partial}{\partial z}, \quad A = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, \quad B = \frac{\partial}{\partial y} + t \frac{\partial}{\partial x} + ty \frac{\partial}{\partial z},$$

and we have that

$$[A, B] = -V, \quad [U, B] = A,$$

the rest of the basic Lie brackets being zero. Thus, G is a connected simply connected nilpotent Lie group. The dual basis of left-invariant 1-forms on G is $\{\omega, \eta, \alpha, \beta\}$, with

$$\omega = dt, \quad \eta = dz - ydx, \quad \alpha = dx - tdy, \quad \beta = dy,$$

and

$$d\omega = d\beta = 0, \quad d\alpha = \beta \wedge \omega, \quad d\eta = \alpha \wedge \beta. \quad (9)$$

From (8), it follows that the lattice Γ of H is invariant under the restriction to \mathbb{Z} of the representation ϕ . This implies that $\Xi = \Gamma \rtimes_{\phi} \mathbb{Z}$ is a lattice in G and $M = \Xi \backslash G$ is a compact nilmanifold.

Now, it is easy to prove that the map

$$M \rightarrow L_{(\bar{\phi}, 1)}, \quad [((x, y, z), t)] \rightarrow [([x, y, z]), t)]$$

is a diffeomorphism and, therefore, $L_{(\bar{\phi}, 1)}$ is a compact nilmanifold.

Remark 3.2. Note that, under the identification between $L_{(\bar{\phi}, 1)}$ and M , the lcs structure of the first kind on $L_{(\bar{\phi}, 1)}$ is just the couple $(\bar{\omega}, \bar{\eta})$, where $\bar{\omega}$ and $\bar{\eta}$ are the 1-forms on M induced by the left-invariant 1-forms ω and η , respectively, on G .

Moreover, we may prove the following result.

Lemma 3.3. $L_{(\bar{\phi},1)}$ is a compact nilmanifold of dimension 4 which admits a lcs structure of the first kind and $b_1(L_{(\bar{\phi},1)}) = 2$.

Proof. Let $H^1(L_{(\bar{\phi},1)}; \mathbb{R})$ be the first de Rham cohomology group of $L_{(\bar{\phi},1)}$. Then, using (9) and Theorem 2.10, we deduce that

$$H^1(L_{(\bar{\phi},1)}; \mathbb{R}) = \langle \bar{\beta}, \bar{\omega} \rangle,$$

where $\bar{\beta}$ is the 1-form on $M \simeq L_{(\bar{\phi},1)}$ induced by the left-invariant 1-form β on G . \square

Theorem 3.4. $L_{(\bar{\phi},1)}$ is a compact nilmanifold of dimension 4 which admits a lcs structure of the first kind; however, $L_{(\bar{\phi},1)}$ does not carry any lcK metric.

Proof. Assume that $L_{(\bar{\phi},1)}$ carries a lcK metric. Then $L_{(\bar{\phi},1)}$ is a compact complex surface. By the Kodaira-Enriques classification, $L_{(\bar{\phi},1)}$ also carries a Kähler metric, since its first Betti number is even by Lemma 3.3 (also, by [8, IV. Theorem 3.1], a complex surface with even first Betti number admits a Kähler metric). Now $L_{(\bar{\phi},1)}$ is a nilmanifold by Lemma 3.3. It is well known (see [14, 31]) that a compact Kähler nilmanifold is diffeomorphic to a torus. Hence $L_{(\bar{\phi},1)}$ should be diffeomorphic to the 4-dimensional torus T^4 , but this is absurd, since $b_1(T^4) = 4$ and $b_1(L_{(\bar{\phi},1)}) = 2$. \square

We also have the following result:

Proposition 3.5. The nilmanifold $L_{(\bar{\phi},1)}$ is not the product of a compact 3-dimensional manifold and a circle.

Proof. By Lemma 3.3, we see that $b_1(L_{(\bar{\phi},1)}) = 2$. Assume $L_{(\bar{\phi},1)}$ is a product $M \times S^1$, where M is a compact 3-dimensional manifold. Since $L_{(\bar{\phi},1)}$ is a compact nilmanifold, $\pi_1(L_{(\bar{\phi},1)})$ is nilpotent and torsion-free. Then $\pi_1(M) \subset \pi_1(L_{(\bar{\phi},1)})$ is also nilpotent and torsion-free. In [44], Mal'tsev showed that for such a group $\pi_1(M)$ there exists a real nilpotent Lie group K such that $P = \pi_1(M) \backslash K$ is a nilmanifold. It is well known that a compact nilmanifold has first Betti number at least 2, hence $b_1(P) \geq 2$. Now P is an aspherical manifold; in this case, $H^*(P; \mathbb{Z})$ is isomorphic to the group cohomology $H^*(\pi_1(M); \mathbb{Z})$ (see for instance [15, page 40]). Hence $b_1(M) \geq 2$. Now M is also an aspherical space with fundamental group $\pi_1(M)$, hence, by the same token, $b_1(\pi_1(M)) \geq 2$. However, by the Künneth formula applied to $L_{(\bar{\phi},1)} = M \times S^1$, we get $b_1(M) = 1$. Alternatively, we could apply [24, Theorem 3] directly to our manifold M . \square

Now, from Theorem 3.4 and Proposition 3.5, we deduce the following result.

Corollary 3.6. There exists a compact, 4-dimensional nilmanifold, not diffeomorphic to the product of a compact 3-manifold and a circle, which has a locally conformal symplectic structure but no locally conformal Kähler metric.

Remark 3.7. The manifold $L_{(\bar{\phi},1)}$ also admits a symplectic structure σ , coming from a left-invariant symplectic structure on G . In terms of the basis of \mathfrak{g}^* given above, $\sigma = \omega \wedge \alpha + \eta \wedge \beta$. $L_{(\bar{\phi},1)}$ can also be endowed with a left-invariant, non-integrable almost complex structure, given in terms of the basis $\{U, V, A, B\}$ of \mathfrak{g} , by $J(U) = A$, $J(V) = B$ and a left-invariant metric g which makes such basis orthonormal. By the Gray-Hervella classification of almost Hermitian manifolds in dimension 4 (see Table II in [28]) there is a line \mathcal{W}_2 corresponding to the almost Kähler case and a line \mathcal{W}_4 corresponding to the complex case, intersecting in the origin, which is the Kähler case. With the almost Hermitian structure (g, J, σ) , $L_{(\bar{\phi},1)}$ lies on the line \mathcal{W}_2 .

As we remarked in the introduction, Bande and Kotschick ([4]) gave a method to construct examples of compact manifolds with lcs structures of the first kind which carry no lcK metrics; we describe it briefly. Suppose that M is an oriented compact manifold of dimension 3. By a result of Martinet, see [47], M admits a contact structure and, then, the product manifold $M \times S^1$ is locally conformal symplectic of the first kind (see Example 2.3). On the other hand, using some results on compact complex surfaces, the authors show that M may be chosen in such a way that the product manifold $M \times S^1$ admits no complex structures (this happens, for instance, if M is hyperbolic) and, in particular, no lcK structures. Moreover, by the results of [25], only for special choices of M (those who fiber over a circle) can $M \times S^1$ have a symplectic structure. Therefore, for suitably chosen M , one obtains examples of locally conformal symplectic structures on $M \times S^1$, with no symplectic structure. We can then ask:

Is there a compact, almost Hermitian lcs 4-manifold, not the product of a 3-manifold and a circle, which admits no complex and no symplectic structure?

Concerning this question, we remark that certain Inoue surfaces, discovered by Belgun (see [13, Theorem 7]), do not carry any lcK metric compatible with the *fixed* complex structure. One may, however, fix the underlying smooth manifold and vary the complex structure. For example, the smooth manifolds underlying Inoue surfaces are all diffeomorphic to a certain solvmanifold, quotient of the completely solvable Lie group G whose Lie algebra is $(0, 12, -13, 23)$. For particular choices of the complex structure, such manifolds *do* carry lcK metrics.

3.2 Dimension 6

In this section, we describe two 6-dimensional examples of lcs nilmanifolds which were announced in lines 2 and 3 of Table 1.

We start by recalling a result of Sawai (see [62, Main Theorem]) which completely characterizes lcK nilmanifolds with left-invariant complex structure. For a description of the Lie algebra of the Heisenberg group, we refer to the proof of Lemma 3.10 below.

Theorem 3.8. *Let (M^{2n}, J) be a non-toral compact nilmanifold with a left-invariant complex structure J . If (M, J) carries a locally conformal Kähler metric, then it is biholomorphic to a quotient of $(\text{Heis}_{2n-1} \times \mathbb{R}, J_0)$, where Heis_{2n-1} is the Heisenberg group of dimension $2n - 1$.*

We consider the 5-dimensional Lie group

$$H = \left\{ \begin{pmatrix} 1 & y & t & w \\ 0 & 1 & x & z \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid x, y, z, t, w \in \mathbb{R} \right\},$$

which is diffeomorphic to \mathbb{R}^5 as a manifold; we denote a point of H by (x, y, z, t, w) .

For the first example, we choose the following basis of left-invariant 1-forms:

$$\alpha = dx, \quad \beta = dy, \quad \gamma = xdx - dz, \quad \delta = dt - ydx \quad \text{and} \quad \eta = dw - ydz + (xy - t)dx$$

with differentials

$$d\alpha = 0, \quad d\beta = 0, \quad d\gamma = 0, \quad d\delta = \alpha \wedge \beta \quad \text{and} \quad d\eta = \alpha \wedge \delta + \beta \wedge \gamma.$$

In particular, (H, η) is a contact manifold. The Lie algebra of H is $\mathfrak{h} = (0, 0, 0, 12, 14 + 23)$. \mathfrak{h} is nilpotent and isomorphic to $L_{5,3}$ (notation from [12]). Hence H is a connected and simply connected nilpotent Lie group.

The subgroup $\Gamma = \{(m, n, p, q, r) \in H \mid m, n, p, q, r \in \mathbb{Z}\} \subset H$ is a lattice. Since η is left-invariant, it descends to a contact form $\bar{\eta}$ on the compact nilmanifold $L = \Gamma \backslash H$. By Example 2.3, the product $M = L \times S^1$ has a lcs structure of the first kind $(\bar{\omega}, \bar{\eta})$, where $\bar{\omega}$ is the angular form on S^1 ; here we are implicitly identifying $\bar{\eta} \in \Omega^1(L)$ with $\pi_1^* \bar{\eta} \in \Omega^1(M)$ and $\bar{\omega} \in \Omega^1(S^1)$ with $\pi_2^* \bar{\omega} \in \Omega^1(M)$.

Theorem 3.9. *M is a 6-dimensional lcs, complex and symplectic nilmanifold which admits no lcK structures (with left-invariant complex structures). Furthermore, M carries no Vaisman metric.*

Proof. M is a nilmanifold, being the quotient of the connected, simply connected nilpotent Lie group $H \times \mathbb{R}$ by the lattice $\Gamma \times \mathbb{Z}$. The Lie algebra of $H \times \mathbb{R}$ is $\mathfrak{h} \oplus \mathbb{R}$, isomorphic to the Lie algebra \mathfrak{h}_9 of [17]; \mathfrak{h}_9 admits a complex structure by [61, Theorem 3.3] (compare also Table 2 below). In particular, M admits a left-invariant complex structure. To show that M is symplectic, it is enough to find a left-invariant symplectic form on $H \times \mathbb{R}$. Consider $\rho := \alpha \wedge \eta + \delta \wedge \gamma + \beta \wedge \omega \in \Omega^2(H \times \mathbb{R})$, with $\omega = dt$ the canonical 1-form on \mathbb{R} . ρ is left-invariant, closed and non-degenerate, hence descends to a symplectic form on M . By Theorem 2.10, $b_1(L) = b_1(\mathfrak{h}) = 3$, hence $b_1(M) = 4$ by the Künneth formula. By Lemma 3.10, a quotient of $\text{Heis}_5 \times \mathbb{R}$ has $b_1 = 5$, hence M is not a quotient of $\text{Heis}_5 \times \mathbb{R}$. By Theorem 3.8, M does not carry any lcK metric (with left-invariant complex structure). If M carried a Vaisman metric, then $b_1(M)$ should be odd by [71, Corollary 3.6]. \square

Lemma 3.10. *A $2n$ -dimensional nilmanifold N , quotient of $\text{Heis}_{2n-1} \times \mathbb{R}$, has $b_1(N) = 2n - 1$.*

Proof. The Lie algebra \mathfrak{heis}_{2n-1} of Heis_{2n-1} is spanned by vectors $\{X_1, Y_1, \dots, X_{n-1}, Y_{n-1}, Z\}$ with brackets $[X_i, Y_i] = Z$ for every $i = 1, \dots, n-1$. If $\{x_1, y_1, \dots, x_{n-1}, y_{n-1}, z\}$ is the dual basis of \mathfrak{heis}_{2n-1}^* , then the Chevalley-Eilenberg complex of the Lie algebra $\mathfrak{heis}_{2n-1}^* \oplus \mathbb{R}$ has differentials

$$dx_i = dy_i = 0 \quad \forall i = 1, \dots, n-1, \quad dw = 0 \quad \text{and} \quad dz = - \sum_{i=1}^{n-1} x_i \wedge y_i,$$

where w generates the \mathbb{R} -factor. This means that $\mathfrak{heis}_{2n-1}^* \oplus \mathbb{R}$ has $b_1 = 2n - 1$. By Theorem 2.10, the same is true for every compact quotient of $\text{Heis}_{2n-1} \times \mathbb{R}$. \square

For the second example, we take

$$\alpha = dx, \quad \beta = dy, \quad \gamma = dz - xdx, \quad \delta = dt - ydx \quad \text{and} \quad \theta = dw - ydz + (xy - t)dx$$

as a basis of left-invariant 1-forms on H ; hence,

$$d\alpha = 0, \quad d\beta = 0, \quad d\gamma = 0, \quad d\delta = \alpha \wedge \beta \quad \text{and} \quad d\theta = \alpha \wedge \delta - \beta \wedge \gamma.$$

As before, θ is a contact form on H . A diffeomorphism $\phi: H \rightarrow H$ will be a strict contactomorphism provided

$$d(w \circ \phi) - dw = (y \circ \phi)d(z \circ \phi) - ydz - ((x \circ \phi)(y \circ \phi) - (t \circ \phi))d(x \circ \phi) + (xy - t)dx. \quad (10)$$

If we assume $y \circ \phi = y + f(x)$, $z \circ \phi = z + g(x) + h(y)$, $t \circ \phi = t + i(x) + j(y) + k(z)$ and $w \circ \phi = w + \ell(x) + m(y) + n(x, y) + o(x, z)$, then the functions

$$\begin{aligned} f(x) &= x & g(x) &= \frac{x}{2} & h(y) &= y & i(x) &= \frac{x}{6} & j(y) &= \frac{y}{2} \\ \ell(x) &= \frac{x^2 - x^3}{3} & m(y) &= \frac{y^2}{2} & n(x, y) &= xy & o(x, z) &= xz \end{aligned}$$

verify (10) and $\phi_s: H \rightarrow H$, mapping (x, y, z, t, w) to

$$\left(x, y + sx, z + sy + \frac{1}{2}s^2x, t + sz + \frac{1}{2}s^2y + \frac{1}{6}s^3x, w + sxz + \frac{1}{2}sy^2 - \frac{1}{3}sx^3 + s^2xy + \frac{1}{3}s^3x^2 \right)$$

is a strict contactomorphism for every $s \in \mathbb{R}$. More precisely:

Lemma 3.11. *In the above situation, $\{\phi_s\}_{s \in \mathbb{R}}$ is a 1-parameter group of strict contactomorphisms of (H, θ) and each ϕ_s is a Lie group isomorphism.*

The subgroup

$$\Gamma = \{(6m, 2n, p, q, r) \in H \mid m, n, p, q, r \in \mathbb{Z}\} \subset H$$

is a lattice and we have $\phi_s(\Gamma) = \Gamma$, for $s \in \mathbb{Z}$. Hence $L := \Gamma \backslash H$ is a compact nilmanifold, θ induces a contact form $\bar{\theta}$ on L and ϕ_1 descends to a strict contactomorphism $\bar{\phi}: L \rightarrow L$. By Example 2.4, the mapping torus $L_{(\bar{\phi}, 1)}$ of L by the pair $(\bar{\phi}, 1)$,

$$L_{(\bar{\phi}, 1)} = \frac{L \times \mathbb{R}}{\sim_{(\bar{\phi}, 1)}}$$

is a 6-dimensional compact lcs manifold of the first kind.

We show that $L_{(\bar{\phi}, 1)}$ has the structure of a compact nilmanifold. In fact, by Lemma 3.11, we obtain a representation $\phi: \mathbb{R} \rightarrow \text{Aut}(H)$, $s \mapsto \phi_s$, hence we can form the semidirect product $G = H \rtimes_{\phi} \mathbb{R}$, whose group structure is given by

$$\begin{pmatrix} x \\ y \\ z \\ t \\ w \\ s \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ z' \\ t' \\ w' \\ s' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' + sx' \\ z + z' + xx' + sy' + \frac{1}{2}s^2x' \\ t + t' + yx' + sz' + \frac{1}{2}s^2y' + \frac{1}{6}s^3x' \\ w + w' + tx' + yz' + s(x'z' + yy' + \frac{1}{2}(y')^2 - \frac{1}{3}(x')^3) + s^2(x'y' + \frac{1}{2}yx') + \frac{1}{3}s^3(x')^2 \\ s + s' \end{pmatrix}$$

A basis for left-invariant 1-forms on G is given by

- $\omega = ds$;
- $\alpha = dx$;
- $\beta = dy - sdx$;
- $\gamma = dz - sdy + (\frac{s^2}{2} - x)dx$;
- $\delta = dt - sdz + \frac{s^2}{2}dy + (xs - y - \frac{s^3}{6})dx$;
- $\eta = dw - ydz - (t - xy)dx$;

with

$$d\omega = 0 = d\alpha, \quad d\beta = -\omega \wedge \alpha, \quad d\gamma = -\omega \wedge \beta, \quad d\delta = -\omega \wedge \gamma + \alpha \wedge \beta \quad \text{and} \quad d\eta = \alpha \wedge \delta - \beta \wedge \gamma.$$

In particular, the Lie algebra \mathfrak{g} of G is isomorphic to the nilpotent Lie algebra $L_{6,22}$ (resp. \mathfrak{h}_{32}) in the notation of [12] (resp. [17]). Hence G is a connected, simply connected nilpotent Lie group. Moreover, arguing as in Section 3.1, $\Xi = \Gamma \rtimes_{\phi} \mathbb{Z} \subset G$ is a lattice, hence $M = \Xi \backslash G$ is a compact nilmanifold, diffeomorphic to the mapping torus $L_{(\bar{\phi}, 1)}$.

Remark 3.12. Note that, under the identification between $L_{(\bar{\phi}, 1)}$ and M , the lcs structure of the first kind on $L_{(\bar{\phi}, 1)}$ is just the pair $(\bar{\omega}, \bar{\eta})$, where $\bar{\omega}$ and $\bar{\eta}$ are the 1-forms on M induced by the left-invariant 1-forms ω and η , respectively, on G .

Theorem 3.13. *M is a compact, complex and lcs nilmanifold which does not admit symplectic, lcK (with left-invariant complex structure) or Vaisman structures.*

Proof. The lcs structure of M is clear from the above discussion. By [61, Theorem 3.3] the Lie algebra \mathfrak{g} of G does not admit any complex structure. Hence M does not admit any lcK structure

with left-invariant complex structure. Next, assume that M admits a symplectic structure. By [31, Lemma 2], M also admits a left-invariant symplectic structure, i.e. \mathfrak{g} is a symplectic Lie algebra. But this is not the case, according to [12, Section 7]. Hence M does not admit any symplectic structure. Clearly $b_1(\mathfrak{g}) = 2$, hence, again by Nomizu Theorem, $b_1(M)$ is also equal to 2. Therefore M does not carry any Vaisman structure by [71, Corollary 3.6]. \square

3.3 A higher dimensional example

In this section we construct, for each $n \geq 3$, a $2n$ -dimensional nilmanifold M_{2n} . For $n = 3$, M_6 is isomorphic to the nilmanifold described in Theorem 3.9. For $n \geq 4$, M_{2n} admits a lcs structure of the first kind, a complex structure, but no symplectic structures, no lcK structures (with left-invariant complex structure) and no Vaisman structures.

Consider the $(2n - 1)$ -dimensional Lie group

$$H_{2n-1} = \left\{ \begin{pmatrix} 1 & x_{n-2} & x_{n-3} & \cdots & x_1 & y_{n-1} & w \\ 0 & 1 & 0 & \cdots & 0 & 0 & y_{n-2} \\ 0 & 0 & 1 & \cdots & 0 & 0 & y_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 & x_{n-1} & y_1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & x_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 \end{pmatrix} \mid x_i, y_i, w \in \mathbb{R}, i = 1, \dots, n-1 \right\}.$$

As a manifold, H_{2n-1} is diffeomorphic to \mathbb{R}^{2n-1} . Taking global coordinates $(x_1, y_1, \dots, x_{n-1}, y_{n-1}, w)$, the multiplication in H_{2n-1} is given by

$$\begin{aligned} & (x_1, y_1, \dots, x_{n-1}, y_{n-1}, w) \cdot (x'_1, y'_1, \dots, x'_{n-1}, y'_{n-1}, w') \\ &= (x_1 + x'_1, y_1 + y'_1 + x_{n-1}x'_{n-1}, x_2 + x'_2, y_2 + y'_2, \dots, x_{n-2} + x'_{n-2}, y_{n-2} + y'_{n-2}, \\ & x_{n-1} + x'_{n-1}, y_{n-1} + y'_{n-1} + x_1x'_{n-1}, w + w' + x_{n-2}y'_{n-2} + x_{n-3}y'_{n-3} + \cdots + x_1y'_1 + y_{n-1}x'_{n-1}). \end{aligned}$$

Moreover, one may compute a basis of left-invariant forms on H_{2n-1} :

- $\alpha_i = -dx_i$, $i = 1, \dots, n-2$;
- $\alpha_{n-1} = dx_{n-1}$;
- $\beta_1 = dy_1 - x_{n-1}dx_{n-1}$;
- $\beta_i = dy_i$, $i = 2, \dots, n-2$;
- $\beta_{n-1} = dy_{n-1} - x_1dx_{n-1}$;
- $\eta = dw - x_1(dy_1 - x_{n-1}dx_{n-1}) - \sum_{i=2}^{n-2} x_i dy_i - y_{n-1}dx_{n-1}$.

These satisfy:

- $d\alpha_i = 0$, $i = 1, \dots, n-1$;
- $d\beta_i = 0$, $i = 1, \dots, n-2$;
- $d\beta_{n-1} = \alpha_1 \wedge \alpha_{n-1}$;
- $d\eta = \sum_{i=1}^{n-1} \alpha_i \wedge \beta_i$.

In particular, \mathfrak{h}_{2n-1} , the Lie algebra of H_{2n-1} , is a nilpotent Lie algebra. For $n = 3$, it is isomorphic to the Lie algebra $L_{5,3}$ which appeared in Section 3.2.

Notice that (H_{2n-1}, η) is a contact manifold. The subgroup $\Gamma_{2n-1} = \{(p_1, q_1, \dots, p_{n-1}, q_{n-1}, r) \in H_{2n-1} \mid p_i, q_i, r \in \mathbb{Z}\} \subset H_{2n-1}$ is a lattice. Since η is left-invariant, it descends to a contact form

$\bar{\eta}$ on the nilmanifold $L_{2n-1} = \Gamma_{2n-1} \backslash H_{2n-1}$. By Example 2.3, the product $M_{2n} = L_{2n-1} \times S^1 = (\Gamma_{2n-1} \times \mathbb{Z}) \backslash (H_{2n-1} \times \mathbb{R})$ has a lcs structure of the first kind $(\bar{\omega}, \bar{\eta})$, where $\bar{\omega}$ is the angular form on S^1 ; as before, we are implicitly identifying $\bar{\eta} \in \Omega^1(L_{2n-1})$ with $\pi_1^* \bar{\eta} \in \Omega^1(M_{2n})$ and $\bar{\omega} \in \Omega^1(S^1)$ with $\pi_2^* \bar{\omega} \in \Omega^1(M_{2n})$.

Set $\mathfrak{g}_{2n} = \mathfrak{h}_{2n-1} \oplus \mathbb{R}$ and let ω be the dual of a generator of the \mathbb{R} -factor. Notice that $d\omega = 0$.

Proposition 3.14. *The Lie algebra \mathfrak{g}_{2n} admits a complex structure.*

Proof. Let $\{X_1, Y_1, \dots, X_{n-1}, Y_{n-1}, Z, T\}$ be the basis of \mathfrak{g}_{2n} dual to $\{\alpha_1, \beta_1, \dots, \alpha_{n-1}, \beta_{n-1}, \eta, \omega\}$. In this basis, the non-zero brackets are

$$[X_i, Y_i] = -Z, \quad i = 1, \dots, n-1 \quad \text{and} \quad [X_1, X_{n-1}] = -Y_{n-1}.$$

We define an endomorphism $J: \mathfrak{g}_{2n} \rightarrow \mathfrak{g}_{2n}$ by setting

$$J(X_1) = -X_{n-1}, \quad J(Y_1) = -Y_{n-1}, \quad J(X_i) = Y_i, \quad i = 2, \dots, n-2, \quad \text{and} \quad J(Z) = -T,$$

and imposing $J^2 = -\text{Id}$. A straightforward computation shows that the Nijenhuis tensor of J vanishes, hence J is a complex structure on \mathfrak{g}_{2n} . \square

Proposition 3.15. *For $n \geq 4$, the Lie algebra \mathfrak{g}_{2n} admits no symplectic structure.*

Proof. Consider the basis $\{\alpha_1, \beta_1, \dots, \alpha_{n-1}, \beta_{n-1}, \eta, \omega\}$ of \mathfrak{g}_{2n}^* and the vector space splitting

$$\mathfrak{g}_{2n}^* = \mathfrak{k}^* \oplus \langle \eta \rangle \oplus \langle \omega \rangle,$$

where $\mathfrak{k}^* = \langle \alpha_1, \beta_1, \dots, \alpha_{n-1}, \beta_{n-1} \rangle$. Then

$$\Lambda^2 \mathfrak{g}_{2n}^* = \Lambda^2 \mathfrak{k}^* \oplus \mathfrak{k}^* \wedge \langle \eta \rangle \oplus \mathfrak{k}^* \wedge \langle \omega \rangle \oplus \langle \omega \rangle \wedge \langle \eta \rangle \quad (11)$$

with

- $d(\Lambda^2 \mathfrak{k}^*) \subset \langle \alpha_1 \wedge \alpha_{n-1} \rangle \wedge \mathfrak{k}^*$
- $d(\mathfrak{k}^* \wedge \langle \eta \rangle) \subset \langle \alpha_1 \wedge \alpha_{n-1} \rangle \wedge \langle \eta \rangle \oplus \mathfrak{k}^* \wedge \langle d\eta \rangle$
- $d(\mathfrak{k}^* \wedge \langle \omega \rangle) \subset \langle \alpha_1 \wedge \alpha_{n-1} \rangle \wedge \langle \omega \rangle$
- $d(\langle \omega \rangle \wedge \langle \eta \rangle) \subset \langle d\eta \rangle \wedge \langle \omega \rangle$.

Given $\sigma \in \Lambda^2 \mathfrak{g}_{2n}^*$, decompose it according to (11) to get

$$\sigma = \sigma_1 + \sigma_2 \wedge \eta + \sigma_3 \wedge \omega + c\omega \wedge \eta.$$

Notice that $d\sigma_1 \in \langle \alpha_1 \wedge \alpha_{n-1} \rangle \wedge \mathfrak{k}^*$, $d(\sigma_2 \wedge \eta) \in \langle \alpha_1 \wedge \alpha_{n-1} \rangle \wedge \langle \eta \rangle \oplus \mathfrak{k}^* \wedge \langle d\eta \rangle$ and $d(\sigma_3 \wedge \omega) \in \langle \alpha_1 \wedge \alpha_{n-1} \rangle \wedge \langle \omega \rangle$. We have $d(c\omega \wedge \eta) = -c\omega \wedge d\eta$; since $d(c\omega \wedge \eta)$ is the only component of $d\sigma$ which can possibly be a multiple of $d\eta \wedge \omega$, $d\sigma = 0$ implies $c = 0$. By the same token, $d(\sigma_2 \wedge \eta)$ is the only component of $d\sigma$ that can possibly lie in $\mathfrak{k}^* \wedge \langle d\eta \rangle$. Hence $d\sigma = 0$ implies $d(\sigma_2 \wedge \eta) = 0$. Write $\sigma_2 = \sum_{i=1}^{n-1} (a_i \alpha_i + b_i \beta_i)$. The only possible non-zero component of $d\sigma$ in $\langle \alpha_1 \wedge \alpha_{n-1} \rangle \wedge \langle \eta \rangle$ comes from a term $d(\beta_{n-1} \wedge \eta)$. Hence, $d\sigma = 0$ implies $b_{n-1} = 0$. We compute

$$\begin{aligned} d \left(\sum_{i=1}^{n-1} a_i \alpha_i \wedge \eta + \sum_{i=1}^{n-2} b_i \beta_i \wedge \eta \right) = \\ - \sum_{i=1}^{n-1} \sum_{j \neq i} a_i \alpha_i \wedge \alpha_j \wedge \beta_j - \sum_{i=1}^{n-2} \sum_{j \neq i} b_i \beta_i \wedge \alpha_j \wedge \beta_j - \sum_{i=1}^{n-2} b_i \beta_i \wedge \alpha_{n-1} \wedge \beta_{n-1}; \end{aligned}$$

we see that $d\sigma = 0$ implies $b_i = 0$ for $i = 1, \dots, n-2$. We are left with $\sigma_2 = \sum_{i=1}^{n-1} a_i \alpha_i$. If $n = 3$,

$$d(\sigma_2 \wedge \eta) = -a_1 \alpha_1 \wedge \alpha_2 \wedge \beta_2 - a_2 \alpha_2 \wedge \alpha_1 \wedge \beta_1 \in \langle \alpha_1 \wedge \alpha_2 \rangle \wedge \mathfrak{k}^*$$

and choosing $a_1 = 0$ and $a_2 = 1$, for instance, we get

$$d(\sigma_2 \wedge \eta) = \alpha_1 \wedge \alpha_2 \wedge \beta_1 = -d(\beta_1 \wedge \beta_2);$$

this gives the symplectic structure $\sigma = \alpha_1 \wedge \omega + \alpha_2 \wedge \eta + \beta_1 \wedge \beta_2$, which was implicitly used in the proof of Theorem 3.9. Hence \mathfrak{g}_6 is symplectic. However, if $n \geq 4$, $\langle \alpha_1 \wedge \alpha_{n-1} \rangle \wedge \mathfrak{k}^* \cap \mathfrak{k}^* \wedge \langle d\eta \rangle = \emptyset$ and if $d(\sigma_2 \wedge \eta)$ has non-zero component in $\mathfrak{k}^* \wedge \langle d\eta \rangle$, then σ is not closed. This forces the a_i to vanish. This implies that every 2-cocycle has rank $< n$, hence \mathfrak{g}_{2n} has no symplectic structure. \square

Theorem 3.16. *For every $n \geq 4$, M_{2n} is a $2n$ -dimensional nilmanifold which is complex, locally conformal symplectic but has no symplectic structure, carries no locally conformal Kähler metric (with left-invariant complex structure) and no Vaisman metric.*

Proof. By construction, $M_{2n} = (\Gamma_{2n-1} \times \mathbb{Z}) \backslash (H_{2n-1} \times \mathbb{R})$, hence it is a nilmanifold. We have already described the lcs structure on M_{2n} . By Proposition 3.14, $H_{2n-1} \times \mathbb{R}$ admits a left-invariant complex structure, which therefore endows M_{2n} with a complex structure. Also, since $n \geq 4$, \mathfrak{g}_{2n} admits no symplectic structure by Proposition 3.15, hence $H_{2n-1} \times \mathbb{R}$ has no left-invariant symplectic structure. Therefore, using Lemma 2 in [31], we deduce that M_{2n} carries no symplectic structures.

One computes readily that $b_1(\mathfrak{g}_{2n}) = 2n-2$; hence, by Nomizu Theorem 2.10, $b_1(M_{2n}) = 2n-2$ and M_{2n} is not a quotient of the Heisenberg group multiplied by \mathbb{R} , according to Lemma 3.10. Again by Theorem 3.8, M_{2n} does not carry any lcK structure (with left invariant complex structure). Finally, since $b_1(M)$ is even M_{2n} carries no Vaisman structure by [71, Corollary 3.6]. \square

Remark 3.17. By Sawai's Theorem 3.8, if M_{2n} carries a lcK metric, then the corresponding complex structure can not be left-invariant. This raises the question of studying nilmanifolds endowed with complex structures that are not left-invariant.

4 Local and global structure of a lcs manifold of the first kind with a compact leaf in its canonical foliation

In this section we present a description of the local and global structure of a compact lcs manifold of the first kind with a compact leaf in its canonical foliation. For this purpose, we use some results on a special class of foliations of codimension 1. We will include basic proofs of them in the next subsection.

4.1 Some general results on a special class of foliations of codimension 1

Let ω be a closed 1-form on a smooth manifold M , with $\omega(p) \neq 0$, for every $p \in M$. The codimension 1 foliation \mathcal{F} , whose characteristic space at $p \in M$ is $\mathcal{F}(p) = \{v \in T_p M \mid \omega(p)(v) = 0\}$, is said to be *transversely parallelizable complete* if there exists a complete vector field U on M such that $\omega(U) = 1$.

Then, we may prove the following results:

Theorem 4.1. *(the local description) Let ω be a closed 1-form on a smooth manifold M such that $\omega(p) \neq 0$, for every $p \in M$. Suppose that L is a compact leaf of the foliation $\mathcal{F} = \{\omega = 0\}$ and that \mathcal{F} is transversely parallelizable complete, U being a complete vector field on M such that $\omega(U) = 1$.*

If $\Psi_L: L \times \mathbb{R} \rightarrow M$ is the restriction to $L \times \mathbb{R}$ of the flow $\Psi: M \times \mathbb{R} \rightarrow M$ of U then there exists $\epsilon > 0$ and an open subset W of M , $L \subseteq W$, such that $\omega|_W$ is an exact 1-form and

$$\Psi_L^\epsilon := (\Psi_L)|_{L \times (-\epsilon, \epsilon)}: L \times (-\epsilon, \epsilon) \rightarrow W$$

is a diffeomorphism. Moreover,

$$(\Psi_L^\epsilon)^*(\omega|_W) = pr_2^*(dt), \quad U|_W \circ \Psi_L^\epsilon = T\Psi_L^\epsilon \circ \frac{\partial}{\partial t}|_{L \times (-\epsilon, \epsilon)} \quad (12)$$

t being the standard coordinate on \mathbb{R} . In addition, the leaves of \mathcal{F} over points of W are of the form $\Psi_t(L)$, with $t \in (-\epsilon, \epsilon)$. In particular, they are contained in W and they are diffeomorphic to L .

Theorem 4.2. (the global description) Under the same hypotheses as in Theorem 4.1 and if, in addition, M is connected then all the leaves of the foliation \mathcal{F} are of the form $\Psi_t(L)$ (with $t \in \mathbb{R}$) and, therefore, diffeomorphic to L . Furthermore, we have two possibilities:

1. If M is not compact then the map $\Psi_L: L \times \mathbb{R} \rightarrow M$ is a diffeomorphism and

$$\Psi_L^*(\omega) = pr_2^*(dt), \quad U \circ \Psi_L = T\Psi_L \circ \frac{\partial}{\partial t}, \quad (13)$$

where $T\Psi_L: T(L \times \mathbb{R}) \rightarrow TM$ is the tangent map of Ψ_L .

2. If M is compact then there exists $c > 0$ such that $\Psi_c: L \rightarrow L$ is a diffeomorphism and the map $\Psi_L: L \times \mathbb{R} \rightarrow M$ induces a diffeomorphism between M and the mapping torus of L by Ψ_c and c .
3. Under the identification between M and $L \times_{(\Psi_c, c)} \mathbb{R}$, U is the vector field on $L \times_{(\Psi_c, c)} \mathbb{R}$ which is induced by the vector field $\frac{\partial}{\partial t}$ on $L \times \mathbb{R}$ and ω is the 1-form on M which is induced by the canonical exact 1-form dt on $L \times \mathbb{R}$. So, if $\pi: M = L \times_{(\Psi_c, c)} \mathbb{R} \rightarrow S^1 = \mathbb{R}/c\mathbb{Z}$ is the canonical projection and τ is the length element of S^1 , we have that $\pi^*(\tau) = \omega$.

Proof. (of Theorem 4.1) Let $p \in L$ be a point. The tangent space to M at p decomposes as $T_p M = T_p L \oplus \langle U(p) \rangle$. Since Ψ is the flow of U , one has

$$(T_{(p,0)}\Psi_L) \left(\frac{\partial}{\partial t} \Big|_{(p,0)} \right) = U(p), \quad (T_{(p,0)}\Psi_L)(X(p)) = X(p)$$

for every vector $X(p) \in T_p L$. Hence $T_{(p,0)}\Psi_L: T_p L \times \mathbb{R} \rightarrow T_p M$ is an isomorphism, thus there exist $\epsilon_p > 0$, an open set $W_p^L \subset L$ with $p \in W_p^L$ and an open set $W_p \subset M$ with $p \in W_p$ such that

$$\Psi|_{W_p^L \times (-\epsilon_p, \epsilon_p)}: W_p^L \times (-\epsilon_p, \epsilon_p) \rightarrow W_p$$

is a diffeomorphism. Clearly $L = \bigcup_{p \in L} W_p^L$, and since L is compact there exist $p_1, \dots, p_k \in L$ such that $L = \bigcup_{i=1}^k W_{p_i}^L$. Set $\epsilon = \min\{\epsilon_{p_1}, \dots, \epsilon_{p_k}\}$ and $W = \Psi(L \times (-\epsilon, \epsilon))$. Then $W \subset M$ is an open set, $L \subset W$ and

$$\Psi_L^\epsilon := \Psi|_{L \times (-\epsilon, \epsilon)}: L \times (-\epsilon, \epsilon) \rightarrow W$$

is a diffeomorphism. On the other hand, we have that

$$\mathcal{L}_U \omega = d(\omega(U)) + i_U(d\omega) = 0.$$

Hence $\Psi_t^* \omega = \omega$ and, as the pullback of $\omega|_W$ to L under the inclusion $L \hookrightarrow W$ is zero, we get $(\Psi_L^\epsilon)^*(\omega|_W) = pr_2^*(dt)$. Furthermore, using that Ψ is the flow of U , we directly deduce the second relation in (12).

Finally, we show that if $t \in (-\epsilon, \epsilon)$ and $p \in L$, then $\Psi_t(L)$ is the leaf L' of \mathcal{F} over $\Psi_t(p)$. It is clear that $\Psi_t(L)$ is a submanifold of M which is diffeomorphic to L . Moreover, since $\Psi_t^* \omega = \omega$, $\Psi_t(L)$ is a connected integral submanifold of M , hence $\Psi_t(L) \subset L'$. Thus $L \subset \Psi_{-t}(L')$, and $\Psi_{-t}(L')$ is an integral submanifold of \mathcal{F} . But this implies $L = \Psi_{-t}(L')$, and $L' = \Psi_t(L)$. \square

Proof. (of Theorem 4.2) We will proceed in three steps.

First step We will see that $\Psi_L: L \times \mathbb{R} \rightarrow M$ is a surjective local diffeomorphism which maps the submanifolds $L \times \{t\}$, with $t \in \mathbb{R}$, to the leaves of the canonical foliation \mathcal{F} of M . In particular, each leaf of \mathcal{F} is of the form $\Psi_t(L)$, $t \in \mathbb{R}$, and it is diffeomorphic to L .

In fact, proceeding as in the proof of Theorem 4.1, we deduce that (13) holds and that if $(p, t) \in L \times \mathbb{R}$ then $\Psi_t(L)$ is the leaf of \mathcal{F} over the point $\Psi_t(p)$. This implies that Ψ_L is a local diffeomorphism. Indeed, if $X \in T_p L$ and $\lambda \in \mathbb{R}$ satisfy

$$0 = (T_{(p,t)} \Psi_L) \left(X + \lambda \frac{\partial}{\partial t} \Big|_t \right) = (T_p \Psi_t)(X) + \lambda U(p)$$

we have that

$$0 = \omega_p((T_p \Psi_t)(X) + \lambda U(p)) = \lambda,$$

(note that $(T_p \Psi_t)(X) \in \mathcal{F}(\Psi_t(p))$). Hence $(T_p \Psi_t)(X) = 0$, which gives $X = 0$.

We claim that $\Psi_L(L \times \mathbb{R}) = M$. Since Ψ_L is a local diffeomorphism, it is clear that $\Psi_L(L \times \mathbb{R})$ is an open subset of M . We will show that $\Psi_L(L \times \mathbb{R})$ is a closed subset of M ; the claim will follow from this and from the assumption that M is connected. Take $p' \in M - \Psi_L(L \times \mathbb{R})$ and denote by L' the leaf of \mathcal{F} over p' . Then there exist $\epsilon' > 0$ and W' an open subset of M such that $L' \subseteq W'$ and

$$\Psi|_{L' \times (-\epsilon', \epsilon')}: L' \times (-\epsilon', \epsilon') \rightarrow W'$$

is a diffeomorphism. We will show that $W' \subset M - \Psi_L(L \times \mathbb{R})$. Indeed, suppose that $q' \in W'$ and $q' = \Psi_t(p)$ for some $p \in L$. Then there exist $r' \in L'$ and $t' \in (-\epsilon', \epsilon')$ such that $q' = \Psi_t(p) = \Psi_{t'}(r')$. Thus $r' = \Psi_{t-t'}(p) \in L' \cap \Psi_{t-t'}(L)$. Since L and L' are leaves of \mathcal{F} , we conclude that $L' = \Psi_{t-t'}(L)$, a contradiction. We have proved so far that $\Psi_L(L \times \mathbb{R}) = M$. This concludes the first step.

Second step We will see that the space of leaves $M/\mathcal{F} = \widetilde{M}$ of the canonical foliation \mathcal{F} is a smooth manifold such that the canonical projection $\pi: M \rightarrow \widetilde{M}$ is a submersion.

Let $p' \in M$ be a point and let L' be the leaf of \mathcal{F} through p' . By Theorem 4.1, there exist an open set $W' \subset M$, with $L' \subset W'$, and $\epsilon' > 0$ such that

$$\Psi|_{L' \times (-\epsilon', \epsilon')}: L' \times (-\epsilon', \epsilon') \rightarrow W'$$

is a diffeomorphism. Take coordinates on some open subset $W'_{L'} \subset L'$ with $p' \in W'_{L'}$. Then $\hat{W}' = \Psi_{L'}(W'_{L'} \times (-\epsilon', \epsilon'))$ is an open subset of M which admits a system of coordinates adapted to the foliation \mathcal{F} , and $p' \in \hat{W}'$. Moreover, if $t', s' \in (-\epsilon', \epsilon')$, $t' \neq s'$, then the plaques $\Psi_{t'}(W'_{L'})$ and $\Psi_{s'}(W'_{L'})$ in \hat{W}' are contained in different leaves of \mathcal{F} . This is precisely the condition needed to ensure that the quotient space $\widetilde{M} = M/\mathcal{F}$ is a smooth manifold and that the canonical projection $\pi: M \rightarrow \widetilde{M}$ is a submersion.

Third step Let L be our compact leaf and let $p \in L$ be a point. Set

$$A_p = \{t \in \mathbb{R} - \{0\} \mid \Psi_t(p) \in L\} \subset \mathbb{R} - \{0\}.$$

We will see that:

- $A_p = \emptyset$ gives the first possibility of the theorem, and
- $A_p \neq \emptyset$ gives the second possibility.

In fact, suppose that $A_p = \emptyset$. Then, since all the leaves of \mathcal{F} are of the form $\Psi_t(L)$ for some $t \in \mathbb{R}$, the map $\Psi_L: \mathbb{R} \times L \rightarrow M$ is injective, hence a diffeomorphism by the first step.

Now, assume that $A_p \neq \emptyset$. We shall see that there exists $c \in \mathbb{R}$, $c > 0$, such that $A_p = c\mathbb{Z}$. In fact, set

$$c = \inf A_p^+, \quad A_p^+ = \{t \in \mathbb{R}^+ \mid \Phi_t(p) \in L\}.$$

Note that $A_p^+ \neq \emptyset$. In addition, from Theorem 4.1 follows that $c > 0$. Furthermore, being L a closed submanifold, we see that $c \in A_p^+$. Therefore $p \in L \cap \Psi_{-c}(L)$ which implies $L = \Psi_{-c}(L)$ and $L = \Psi_c(L)$. Hence, if $k \in \mathbb{Z}$, $ck \in A_p$ and $c\mathbb{Z} \subset A_p$. Conversely, if $t \in A_p$, there exists $k \in \mathbb{Z}$ such that $ck \leq t < c(k+1)$. If $t > ck$, we have $0 < t - ck < c$ and $t - ck \in A_p^+$, which contradicts the fact that c is the infimum of A_p^+ . Hence $t = ck$ and $A_p = c\mathbb{Z}$.

To conclude, we will show that $\Psi_L: L \times \mathbb{R} \rightarrow M$ induces a diffeomorphism between the manifold $L \times_{(\Psi_c, c)} \mathbb{R}$ and M . For this, it is sufficient to prove that if (y, t) and (y', t') in $L \times \mathbb{R}$, then $\Psi_L(y, t) = \Psi_L(y', t')$ if and only if there exists $k \in \mathbb{Z}$ such that

$$y = \Psi_{ck}(y') \quad \text{and} \quad t' = t + ck. \quad (14)$$

Clearly, if (14) holds, then $\Psi_L(y, t) = \Psi_L(y', t')$. Conversely, suppose $\Psi_L(y, t) = \Psi_L(y', t')$ and suppose $t' \geq t$. Then $y = \Psi_{t'-t}(y') \in L \cap \Psi_{t'-t}(L)$, from which follows as usual $L = \Psi_{t'-t}(L)$. In particular, $\Psi_{t'-t}(p) \in L$, so that $t' - t \in A_p$. Therefore, there exists $k \in \mathbb{Z}$ such that $t' = t + ck$ and from this we see that $y = \Psi_{ck}(y')$. This concludes the proof of the theorem. \square

4.2 The particular case of a lcs manifold of the first kind with a compact leaf in its canonical foliation

In this section, we will consider the particular case when M has a lcs structure of the first kind and ω is the Lee 1-form of M .

We recall that if L is a leaf of the canonical foliation $\mathcal{F} = \{\omega = 0\}$, η is the anti-Lee 1-form of M and $i: L \rightarrow M$ is the canonical inclusion then $\eta_L = i^*(\eta)$ is a contact 1-form on L . Thus:

- The product manifold $L \times (-\epsilon, \epsilon)$, with $\epsilon > 0$, admits a canonical gcs structure of the first kind and
- If L is compact, $\phi: L \rightarrow L$ is a strict contactomorphism and $c > 0$ then the mapping torus $L \times_{(\phi, c)} \mathbb{R}$ of L by ϕ and c is a compact lcs manifold of the first kind

(see Section 2.1 for more details).

Now, we will introduce two natural definitions which will be useful in the sequel. The first one is well known (see, for instance, [42]).

Definition 4.3. Let η_1 and η_2 be contact forms on the manifolds M_1 and M_2 respectively. A diffeomorphism $\Psi: M_1 \rightarrow M_2$ is said to be a *strict contactomorphism* if $\Psi^*\eta_2 = \eta_1$.

Definition 4.4. Let (ω_1, η_1) and (ω_2, η_2) , be lcs structures of the first kind on manifolds M_1 and M_2 respectively. A diffeomorphism $\Psi: M_1 \rightarrow M_2$ is said to be a *lcs morphism of the first kind* if $\Psi^*\omega_2 = \omega_1$ and $\Psi^*\eta_2 = \eta_1$.

Remark 4.5. Ψ being a lcs morphism of the first kind implies that $\Psi^*\Phi_2 = \Phi_1$, where $\Phi_i = d\eta_i + \eta_i \wedge \omega_i$, $i = 1, 2$.

We prove the two following results

Theorem 4.6. (the local description) Let (ω, η) be a lcs structure of the first kind on a manifold M and let U be the anti-Lee vector field of M . Suppose that U is complete and that L is a compact leaf of the canonical foliation $\mathcal{F} = \{\omega = 0\}$ on M . If $\Psi_L: L \times \mathbb{R} \rightarrow M$ is the restriction to $L \times \mathbb{R}$ of the flow $\Psi: M \times \mathbb{R} \rightarrow M$ of U then, for every point $p \in L$, there exist an open subset $W \subseteq M$, $L \subseteq W$, and a positive real number $\epsilon > 0$ such that

$$\Psi_L^\epsilon := \Psi|_{L \times (-\epsilon, \epsilon)}: L \times (-\epsilon, \epsilon) \rightarrow W$$

is an isomorphism between the gcs manifolds $L \times (-\epsilon, \epsilon)$ and W . Moreover, the leaves of \mathcal{F} over points of W are of the form $\Psi_t(L)$, with $t \in (-\epsilon, \epsilon)$. In particular, they are contained in W and they are strict contactomorphic to L .

Theorem 4.7. (the global description) If, in addition to the hypotheses of Theorem 4.6, M is connected, then all the leaves of the canonical foliation are of the form $\Psi_t(L)$ (with $t \in \mathbb{R}$) and, therefore, they are strict contactomorphic to L . Furthermore, we have the two following possibilities:

1. If M is not compact then the map $\Psi_L: L \times \mathbb{R} \rightarrow M$ is a gcs isomorphism between the gcs manifolds $L \times \mathbb{R}$ and M .
2. If M is compact then there exists $c > 0$ such that $\Psi_c: L \rightarrow L$ is a strict contactomorphism and Ψ_L induces a lcs isomorphism of the first kind between M and the mapping torus of L by Ψ_c and c .

In order to prove Theorems 4.6 and 4.7, we will use the following result:

Proposition 4.8. Under the same hypotheses as in Theorem 4.6, we have that

$$\Psi_L^*(\eta) = pr_1^*(\eta_L),$$

where $pr_1: L \times \mathbb{R} \rightarrow L$ is the canonical projection on the first factor and η_L is the contact 1-form on L .

Proof. It is clear that

$$\mathcal{L}_U \eta = d(\eta(U)) + i_U(d\eta) = 0,$$

and, thus,

$$\Psi_t^* \eta = \eta, \quad \text{for every } t \in \mathbb{R}. \quad (15)$$

In addition, using that $\eta(U) = 0$, we also have that

$$(\Psi_L^* \eta) \left(\frac{\partial}{\partial t} \right) = 0. \quad (16)$$

Therefore, from (15) and (16), we conclude that

$$\Psi_L^* \eta = pr_1^*(\eta_L).$$

□

Proof. (of Theorem 4.6) It follows using Theorem 4.1 and Proposition 4.8. □

Proof. (of Theorem 4.7) It follows using Theorem 4.2 and Proposition 4.8. □

The first possibility of Theorem 4.7 means that M is symplectomorphic to the standard symplectization of the leaf. In the compact case, Theorem 4.7 tells us that a manifold endowed with a locally conformal symplectic structure of the first kind (ω, η) fibres over a circle, that η pulls back to a contact form on the fibre and that the gluing map is a strict contactomorphism. This displays the similarity with the result of Banyaga. The main difference between the two results is that Banyaga restricts to the compact case and needs to modify the foliation \mathcal{F} to a \mathcal{C}^∞ -near foliation \mathcal{F}' which he proves to be a fibration, while we work directly with \mathcal{F} . In our case, however, we must assume the completeness of U (which is automatic if our manifold is compact) as well as the existence of a compact leaf of \mathcal{F} .

Remark 4.9. A similar issue appears in the context of *cosymplectic manifolds*. Recall that a manifold M^{2n+1} is cosymplectic (in the sense of Libermann, see [41]) if there exist $\alpha \in \Omega^1(M)$ and $\beta \in \Omega^2(M)$ such that $d\alpha = 0 = d\beta$ and $\alpha \wedge \beta^n \neq 0$. It was proven by Li in [40] that, in the compact case, a cosymplectic manifold corresponds to a mapping torus of a symplectic manifold and a symplectomorphism. In particular, such manifolds fibre over S^1 with fibre a symplectic manifold. However, in order to obtain this result (whose proof is similar in spirit to that of Banyaga for the lcs case) one needs to perturb the foliation $\{\alpha = 0\}$, so that the original cosymplectic structure is destroyed. Both the result of Banyaga and that of Li rely on a theorem of Tischler [64], which asserts that a compact manifold with a closed and nowhere zero 1-form fibres over the circle S^1 . Recently, Guillemin, Miranda and Pires (see [30]) have proven a result similar to our Theorem 4.7 in the context of compact cosymplectic manifolds, working directly with the foliation $\{\alpha = 0\}$.

4.3 A Martinet-type result for lcs structures of the first kind

As another application of the results in Section 4.1, we obtain a Martinet-type result about the existence of lcs structures of the first kind on a certain type of oriented compact manifolds of dimension 4.

First of all, we recall the classical result of Martinet [47]: if L is an oriented closed manifold of dimension 3 then there exists a contact 1-form on L . There exist some equivariant versions of this result:

Theorem 4.10 ([36, 54]). *Let L be an oriented closed manifold of dimension 3 and an action of S^1 on L which preserves the orientation. Then there exists a S^1 -invariant contact 1-form on L .*

Theorem 4.11 ([16]). *Let L be an oriented closed manifold of dimension 3 and suppose that a finite group Γ of prime order acts on L preserving the orientation. Then there exists a Γ -invariant contact 1-form on L .*

Now, let M be an oriented connected manifold of dimension 4, ω a closed 1-form on M without singularities and let L be a compact leaf of the foliation $\mathcal{F} = \{\omega = 0\}$.

If M is not compact then, using Theorem 4.2, it directly follows that M admits a gcs structure of the first kind.

Next, suppose that M is compact. Using again Theorem 4.2, we deduce that the global structure of M is completely determined by L , a real number $c > 0$ and a diffeomorphism $\phi: L \rightarrow L$. In fact, if U is a vector field on M such that $\omega(U) = 1$ and $\Psi_L: L \times \mathbb{R} \rightarrow M$ is the restriction to $L \times \mathbb{R}$ of the flow of U then $\phi = \Psi_c$ and Ψ_L induces a diffeomorphism between $L_{(\phi, c)} = (L \times \mathbb{R}) / \sim_{(\phi, c)}$ and M .

On the other hand, since M is orientable, we can choose a volume form ν on $L \times \mathbb{R}$ which is invariant under the transformation

$$L \times \mathbb{R} \rightarrow L \times \mathbb{R}, \quad (x, t) \rightarrow (\phi(x), t - c).$$

Thus, for every $t \in \mathbb{R}$, the 3-form $i(\frac{\partial}{\partial t})\nu$ on $L \times \mathbb{R}$ induces a volume form ν_t on L in such a way that:

- the volume form ν_t is ϕ -invariant and
- any two of these volume forms define the same orientation on L .

Using the previous facts and Theorems 4.2, 4.10 and 4.11, we conclude

Corollary 4.12. *Let M be an oriented connected manifold of dimension 4, ω a closed 1-form on M without singularities and L a compact leaf of the foliation $\mathcal{F} = \{\omega = 0\}$.*

1. *If M is not compact then it admits a gcs structure of the first kind. The structure is globally conformal to the symplectization of a leaf.*
2. *If M is compact then M may be identified with a mapping torus of L by a real number $c > 0$ and a diffeomorphism $\phi: L \rightarrow L$. Moreover:*
 - a) *If there exists an action $\psi: S^1 \times L \rightarrow L$ which preserves the orientation induced on L and $\phi = \psi_\lambda$, for some $\lambda \in S^1$, then M admits a lcs structure of the first kind.*
 - b) *If $\phi: L \rightarrow L$ preserves the orientation induced on L , the discrete subgroup of transformations of M*

$$\Gamma = \{\phi^k \mid k \in \mathbb{Z}\}$$

is finite and its order is prime, then M also admits a lcs structure of the first kind.

5 Locally conformal symplectic Lie algebras

5.1 Lcs structures on Lie algebras

In this section we describe locally conformal symplectic structures on Lie algebras.

Definition 5.1. Let \mathfrak{g} be a real Lie algebra of dimension $2n$ ($n \geq 2$). A *locally conformal symplectic (lcs) structure* on \mathfrak{g} consists of:

- $\Phi \in \Lambda^2 \mathfrak{g}^*$, non-degenerate, i.e. $\Phi^n \neq 0$;
- $\omega \in \mathfrak{g}^*$, with $d\omega = 0$, such that $d\Phi = \omega \wedge \Phi$.

Here d is the Chevalley-Eilenberg differential on $\Lambda^\bullet \mathfrak{g}^*$. Locally conformal symplectic Lie algebras (along with contact Lie algebras) have been considered in [35], as a special instance of algebraic Jacobi structures.

Let $(\mathfrak{g}, \Phi, \omega)$ be a lcs Lie algebra. Since Φ is non-degenerate, it defines an isomorphism $\mathfrak{g} \rightarrow \mathfrak{g}^*$, $X \mapsto \iota_X \Phi$.

The *automorphisms* of the lcs structure (Φ, ω) , denoted \mathfrak{g}_Φ , are the elements of \mathfrak{g} which preserve the 2-form Φ , that is

$$\mathfrak{g}_\Phi = \{X \in \mathfrak{g} \mid L_X \Phi = 0\}.$$

$\mathfrak{g}_\Phi \subset \mathfrak{g}$ is a Lie subalgebra of \mathfrak{g} . Let $\ell: \mathfrak{g} \rightarrow \mathbb{R}$ be the map which sends $X \in \mathfrak{g}$ to $\omega(X) \in \mathbb{R}$. Viewing \mathbb{R} as an abelian Lie algebra, the closedness of ω implies that ℓ is a morphism of Lie algebras. Thus, the restriction of ℓ to \mathfrak{g}_Φ also is a Lie algebra morphism known as *Lee morphism*. The image of the Lee morphism is 1-dimensional; hence ℓ is surjective, if it is non-zero.

Definition 5.2. The lcs Lie algebra $(\mathfrak{g}, \Phi, \omega)$ is said to be *of the first kind* if ℓ is surjective; *of the second kind* if it is zero.

In this paper we will deal with lcs Lie algebras of the first kind. Let $(\mathfrak{g}, \Phi, \omega)$ be a lcs Lie algebra of the first kind. Pick a vector $U \in \mathfrak{g}_\Phi$ such that $\ell(U) = 1$. Define $\eta \in \mathfrak{g}^*$ by the equation $\eta = -\iota_U \Phi$; clearly $U \in \ker(\eta)$. Also, define $V \in \mathfrak{g}$ by $\omega = \iota_V \Phi$; notice that $V \in \ker(\omega)$ and that $\iota_V \eta = 1$. Since $U \in \mathfrak{g}_\Phi$, $\mathcal{L}_U \Phi = 0$ and hence $d\iota_U \Phi = -\iota_U d\Phi$, which implies

$$d\eta = -d\iota_U \Phi = \iota_U d\Phi = \iota_U(\omega \wedge \Phi) = \Phi + \omega \wedge \eta.$$

Thus $\Phi = d\eta - \omega \wedge \eta$. Easy computations show that $\iota_U d\eta = \iota_V d\eta = 0$. Therefore $d\eta \in \Lambda^2 \mathfrak{g}^*$ has two vectors in its kernel, and can not have maximal rank n . We compute

$$0 \neq \Phi^n = (d\eta + \eta \wedge \omega)^n = n(d\eta)^{n-1} \wedge \eta \wedge \omega.$$

But then $d\eta$ has rank $2n - 2$ and η behaves like a contact form on the ideal $\ker(\omega)$, which has dimension $2n - 1$.

To sum up, we obtain an algebraic analogue to Proposition 2.2: a lcs structure of the first kind on a Lie algebra \mathfrak{g} of dimension $2n$ is completely determined by two 1-forms $\omega, \eta \in \mathfrak{g}^*$ such that

$$d\omega = 0, \quad \text{rank}(d\eta) < 2n \quad \text{and} \quad \omega \wedge \eta \wedge (d\eta)^{n-1} \neq 0. \quad (17)$$

From now on we will use (ω, η) to denote a lcs structure of the first kind on the Lie algebra \mathfrak{g} . Clearly, $\Phi = d\eta - \omega \wedge \eta$. By (17) there exist $U, V \in \mathfrak{g}$, the *anti-Lee* and *Lee* vectors, characterized by the conditions

$$\begin{aligned} \omega(U) &= 1, & \eta(U) &= 0, & \iota_U d\eta &= 0, \\ \omega(V) &= 0, & \eta(V) &= 1, & \iota_V d\eta &= 0. \end{aligned}$$

Note that the previous conditions imply that

$$i_{[U,V]}\omega = 0, \quad i_{[U,V]}\eta = 0 \quad \text{and} \quad i_{[U,V]}d\eta = 0,$$

and, therefore,

$$[U, V] = 0. \quad (18)$$

Let now $(\mathfrak{g}, \Phi, \omega)$ be a lcs Lie algebra. Since ω is a closed 1-form, we can perform the construction of Example 2.6. In particular, Φ is a 2-cocycle for $d_\omega: C^2(\mathfrak{g}; W_\omega) \rightarrow C^3(\mathfrak{g}; W_\omega)$, hence it defines a cohomology class $[\Phi] \in H^2(\mathfrak{g}; W_\omega)$. We call (Φ, ω) *exact* if $[\Phi] = 0$, *non-exact* otherwise. Assume that the lcs structure is of the first kind. Then there exists $\eta \in \mathfrak{g}^*$ such that $\Phi = d\eta - \omega \wedge \eta$, hence $\Phi = d_\omega \eta$ and Φ is a coboundary. We obtain:

Proposition 5.3. *Let \mathfrak{g} be a Lie algebra endowed with a lcs structure of the first kind (Φ, ω) . Then (Φ, ω) is exact.*

The converse is in general false, as the following example shows:

Example 5.4. Consider the 4-dimensional solvable Lie algebra $\mathfrak{g} = (12 + 34, 0, -23, 0)$, isomorphic to the Lie algebra $\mathfrak{d}_{4,1}$ of the list contained in [60, Proposition 2.1]. Consider the lcs structure on \mathfrak{g} obtained by taking $\Phi = 2e^{12} + e^{34}$ and $\omega = e^2$. One checks that $\Phi = d_\omega \eta$, with $\eta = e^1$, hence the structure is exact. However, it is not of the first kind. Indeed, the only automorphisms of (Φ, ω) are of the form ae_1 , with $a \in \mathbb{R}$; all of them are sent to zero by the Lee morphism. If G denotes the unique simply connected solvable Lie group with Lie algebra \mathfrak{g} , then G is endowed with an exact lcs structure which is not of the first kind. Notice that G is not compact.

Consider an exact lcs Lie algebra $(\mathfrak{g}, \Phi, \omega)$ of dimension $2n$ and write $\Phi = d\eta - \omega \wedge \eta$, for some $\eta \in \mathfrak{g}^*$. Then we have

$$0 \neq \Phi^n = (d\eta - \omega \wedge \eta)^n = (d\eta)^n + n(d\eta)^{n-1} \wedge \eta \wedge \omega, \quad (19)$$

hence $2n - 2 \leq \text{rank}(d\eta) \leq 2n$. If the rank of $d\eta$ is $2n - 2$, then (Φ, ω) is of the first kind by the above construction. On the other hand, if the rank of $d\eta$ is $2n$, the fact that $\wedge^{2n} \mathfrak{g}^*$ is 1-dimensional, together with (19), shows that Φ^n must be a multiple of $(d\eta)^n$. This implies that the volume form is exact.

Recall that a Lie algebra \mathfrak{g} is *unimodular* if $\text{tr}(\text{ad}_X) = 0$ for each $X \in \mathfrak{g}$. An equivalent characterization is given, in terms of the Chevalley-Eilenberg complex $(\Lambda^\bullet \mathfrak{g}^*, d)$, by the condition $H^n(\mathfrak{g}; \mathbb{R}) \neq 0$, where $n = \dim \mathfrak{g}$. Hence, on a unimodular Lie algebra, the volume form can not be exact. Let G be the unique connected simply connected Lie group with Lie algebra \mathfrak{g} . By [53, Lemma 6.2], if G admits a discrete subgroup Γ with compact quotient, then \mathfrak{g} is unimodular. In particular, if \mathfrak{g} is not unimodular, then G does not admit any compact quotient.

Putting all these considerations together, we obtain

Proposition 5.5. *Let \mathfrak{g} be a unimodular Lie algebra endowed with an exact lcs structure (Φ, ω) . Then (Φ, ω) is of the first kind.*

In particular, on unimodular Lie algebras a lcs structure is exact if and only if it is of the first kind. Notice that the Lie algebra of Example 5.4 is not unimodular.

5.2 Contact structures on Lie algebras

Definition 5.6. Let \mathfrak{h} be a real Lie algebra of dimension $2n - 1$. A *contact structure* on \mathfrak{h} a 1-form $\theta \in \mathfrak{h}^*$ such that

$$(d\theta)^{n-1} \wedge \theta \neq 0.$$

Again d denotes the Chevalley-Eilenberg differential on $\Lambda^\bullet \mathfrak{h}^*$. If (\mathfrak{h}, θ) is a contact Lie algebra then there exists a unique vector $R \in \mathfrak{h}$, the *Reeb vector*, which is characterized by the conditions

$$i_R(d\theta) = 0 \quad \text{and} \quad i_R(\theta) = 1.$$

For a contact Lie algebra (\mathfrak{h}, θ) with Reeb vector R , either the center of \mathfrak{h} , $\mathcal{Z}(\mathfrak{h})$, is trivial or $\mathcal{Z}(\mathfrak{h}) = \langle R \rangle$. Contact Lie algebras with non-trivial center are in 1-1 correspondence with a particular class of central extensions of symplectic Lie algebras. In fact, if $\sigma \in \Lambda^2 \mathfrak{s}^*$ is a symplectic structure on a Lie algebra \mathfrak{s} of dimension $2n - 2$, that is, $\sigma^{n-1} \neq 0$ and $d\sigma = 0$, then one may consider the central extension $\mathfrak{h} = \mathbb{R} \odot_\sigma \mathfrak{s}$ of \mathfrak{s} by the 2-cocycle σ . If $\theta \in \mathfrak{h}^*$ is the 1-form on \mathfrak{h} given by

$$\theta(a, X) = a,$$

we have that (\mathfrak{h}, θ) is a contact Lie algebra with Reeb vector $R = (1, 0) \in \mathfrak{h}$. The converse is also true. Namely, if \mathfrak{h} is a contact Lie algebra with Reeb vector R such that $\mathcal{Z}(\mathfrak{h}) = \langle R \rangle$ then the quotient vector space $\mathfrak{s} = \mathfrak{h}/\langle R \rangle$ is a symplectic Lie algebra and \mathfrak{h} is the central extension of \mathfrak{s} by the symplectic structure (for more details, see [20]).

We consider next derivations of contact and symplectic Lie algebras:

Definition 5.7. Let (\mathfrak{h}, θ) be a contact Lie algebra and let $D \in \text{Der}(\mathfrak{h})$ be a derivation. D is called a *contact derivation* if $D^*\theta = 0$. Let (\mathfrak{s}, σ) be a symplectic Lie algebra and let $D \in \text{Der}(\mathfrak{s})$ be a derivation. D is called a *symplectic derivation* if $\sigma(DX, Y) + \sigma(X, DY) = 0$ for every $X, Y \in \mathfrak{s}$.

In order to describe contact derivations on (\mathfrak{h}, θ) , we assume that \mathfrak{h} is the central extension of a symplectic Lie algebra.

Proposition 5.8. *Let (\mathfrak{s}, σ) be a symplectic Lie algebra. There exists a one-to-one correspondence between symplectic derivations in \mathfrak{s} and contact derivations in the central extension $\mathfrak{h} = \mathbb{R} \odot_{\sigma} \mathfrak{s}$. In fact, the correspondence is given as follows. If $D_{\mathfrak{s}}: \mathfrak{s} \rightarrow \mathfrak{s}$ is a symplectic derivation in \mathfrak{s} then D is a contact derivation in \mathfrak{h} , where D is defined by*

$$D(a, X) = (0, D_{\mathfrak{s}}X), \quad \text{for } (a, X) \in \mathfrak{h}. \quad (20)$$

Proof. Let D be a contact derivation of $\mathfrak{h} = \mathbb{R} \odot_{\sigma} \mathfrak{s}$. Then, the condition $D^*(1, 0) = 0$ implies that

$$D(a, X) = (0, aZ + D_{\mathfrak{s}}X), \quad \text{for } (a, X) \in \mathfrak{h}$$

where Z is a fixed vector of \mathfrak{s} and $D_{\mathfrak{s}}: \mathfrak{s} \rightarrow \mathfrak{s}$ is a linear map. By (7), $[(a, 0), (0, X)]_{\mathfrak{h}} = 0$ for every $X \in \mathfrak{s}$, and thus

$$0 = D[(1, 0), (0, Y)]_{\mathfrak{h}} = [D(1, 0), (0, Y)]_{\mathfrak{h}} = (\sigma(Z, Y), [Z, Y]);$$

from this we deduce that $Z = 0$. Thus, $D(a, X) = (0, D_{\mathfrak{s}}X)$. On the other hand, using that

$$D[(0, X), (0, Y)]_{\mathfrak{h}} = [D(0, X), (0, Y)]_{\mathfrak{h}} + [(0, X), D(0, Y)]_{\mathfrak{h}}, \quad \text{for } X, Y \in \mathfrak{s}$$

we conclude that $D_{\mathfrak{s}}$ is a symplectic derivation of the symplectic Lie algebra (\mathfrak{s}, σ) .

Conversely, suppose that $D_{\mathfrak{s}}: \mathfrak{s} \rightarrow \mathfrak{s}$ is a symplectic derivation of the symplectic algebra \mathfrak{s} and let $D: \mathfrak{h} \rightarrow \mathfrak{h}$ be the linear map given by (20). Then, a direct computation, proves that D is a contact derivation of \mathfrak{h} . \square

5.3 A correspondence between contact Lie algebras and lcs Lie algebras of the first kind

Here we prove that there is a 1-1 correspondence between contact Lie algebras in dimension $2n - 1$ endowed with a contact derivation and lcs Lie algebras of the first kind in dimension $2n$.

Theorem 5.9. *Let $(\mathfrak{g}, \omega, \eta)$ be a lcs Lie algebra of the first kind of dimension $2n$. Set $\mathfrak{h} = \ker(\omega)$ and let θ be the restriction of η to \mathfrak{h} . Then (\mathfrak{h}, θ) is a contact Lie algebra, endowed with a contact derivation D and \mathfrak{g} is isomorphic to the semidirect product $\mathfrak{h} \rtimes_D \mathbb{R}$. In fact, D is induced by the inner derivation $\text{ad}_U: \mathfrak{g} \rightarrow \mathfrak{g}$. Conversely, let (\mathfrak{h}, θ) be a contact Lie algebra and let D be a contact derivation of \mathfrak{h} . Then $\mathfrak{g} = \mathfrak{h} \rtimes_D \mathbb{R}$ is endowed with a lcs structure of the first kind.*

Proof. If $X, Y \in \mathfrak{g}$, then

$$\omega([X, Y]) = -d\omega(X, Y) = 0,$$

since ω is closed. This means that the subalgebra $[\mathfrak{g}, \mathfrak{g}]$ is contained in \mathfrak{h} , and \mathfrak{h} is an ideal of \mathfrak{g} . Let U be the anti-Lee vector of the lcs structure (ω, η) on \mathfrak{g} and denote by θ the restriction of η to \mathfrak{h} . Using that $\omega \wedge \eta \wedge (d\theta)^{n-1} \neq 0$ and the fact that $i_U \eta = 0$ and $i_U d\eta = 0$, we conclude that (\mathfrak{h}, θ) is a contact Lie algebra. Define a linear map D on \mathfrak{h} by

$$D(X) = \text{ad}_U(X). \quad (21)$$

Since \mathfrak{h} contains the commutator $[\mathfrak{g}, \mathfrak{g}]$, indeed $D: \mathfrak{h} \rightarrow \mathfrak{h}$, and D is derivation of \mathfrak{h} by the Jacobi identity in \mathfrak{g} . Furthermore, D is a contact derivation of (\mathfrak{h}, θ) :

$$(D^*\theta)(X) = \theta(D(X)) = \theta([U, X]) = \eta([U, X]) = -d\eta(U, X) = -i_U d\eta(X) = 0.$$

Consider the linear isomorphism $\varphi: \mathfrak{g} \rightarrow \mathfrak{h} \rtimes_D \mathbb{R}$ given by

$$\varphi(X) = (X - \omega(X)U, \omega(X)), \quad \text{for } X \in \mathfrak{g}.$$

Note that

$$\varphi^{-1}(X, a) = aU + X, \quad \text{for } (X, a) \in \mathfrak{h} \rtimes_D \mathbb{R}.$$

Therefore, from (21), we conclude that φ is a Lie algebra isomorphism between \mathfrak{g} and $\mathfrak{h} \rtimes_D \mathbb{R}$.

Conversely, let us start with a contact Lie algebra (\mathfrak{h}, θ) of dimension $2n - 1$ and a contact derivation D . Set $\mathfrak{g} = \mathfrak{h} \rtimes_D \mathbb{R}$ and write a vector in \mathfrak{g} as (X, a) , with $a \in \mathbb{R}$. Use (5) to define a Lie algebra structure on \mathfrak{g} . Take $U = (0, 1) \in \mathfrak{g}$. Then,

$$[U, (X, 0)]_{\mathfrak{g}} = (D(X), 0),$$

and D can be identified with the adjoint action of U on \mathfrak{g} . Define $\omega \in \mathfrak{g}^*$ by $\omega|_{\mathfrak{h}} = 0$, $\omega(U) = 1$, and let $\eta \in \mathfrak{g}^*$ be the extension of the contact form θ to \mathfrak{g} obtained by setting $\eta(U) = 0$. To prove that \mathfrak{g} is lcs of the first kind, it is enough to show that $d\omega = 0$ and that $\omega \wedge \eta \wedge (d\eta)^{n-1} \neq 0$. Taking vectors $(X, a), (Y, b) \in \mathfrak{g}$, we have

$$d\omega((X, a), (Y, b)) = -\omega([(X, a), (Y, b)]_{\mathfrak{g}}) = -\omega((aD(Y) - bD(X) + [X, Y]_{\mathfrak{h}}, 0)) = 0,$$

since $\omega|_{\mathfrak{h}} = 0$. We are left with showing that the rank of $d\eta$ is $2n - 2$. One has

$$\begin{aligned} \iota_U d\eta((X, a)) &= d\eta((0, 1), (X, a)) = -\eta([(0, 1), (X, a)]_{\mathfrak{g}}) = -\eta((D(X), 0)) = -\theta(D(X)) = \\ &= -(D^*\theta)(X) = 0, \end{aligned}$$

since D is a contact derivation. This shows that $d\eta$ has a kernel, hence its rank cannot be $2n$. On the other hand, $d\eta = d\theta$ on $\mathfrak{h} = \ker(\omega)$, hence the rank of $d\eta$ is indeed $2n - 2$. Moreover, if V is the Reeb vector of \mathfrak{h} then

$$\iota_V \iota_U (\omega \wedge \eta \wedge (d\eta)^{n-1}) = \iota_V (\eta \wedge (d\eta)^{n-1}) = (d\eta)^{n-1} \neq 0$$

which implies that

$$\omega \wedge \eta \wedge (d\eta)^{n-1} \neq 0.$$

The lcs structure of the first kind is then obtained by setting $\Phi = d\eta - \omega \wedge \eta$. □

Remark 5.10. Theorem 5.9 can be interpreted as an algebraic analogue of Theorem 4.7.

5.4 Symplectic Lie algebras and a particular class of lcs Lie algebras of the first kind

In this section, we will consider a special class of lcs Lie algebras of the first kind, those with central Lee vector. We will see that they are closely related with symplectic Lie algebras. In fact, we have the following result

Theorem 5.11. *There exists a one-to-one correspondence between lcs Lie algebras of the first kind $(\mathfrak{g}, \omega, \eta)$ of dimension $2n + 2$ with central Lee vector and symplectic Lie algebras (\mathfrak{s}, σ) of dimension $2n$ endowed with a symplectic derivation. In fact, this correspondence is defined as follows. Let (\mathfrak{s}, σ) be a symplectic Lie algebra and $D_{\mathfrak{s}}: \mathfrak{s} \rightarrow \mathfrak{s}$ a symplectic derivation in \mathfrak{s} . On the vector space $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{s} \oplus \mathbb{R}$ we can consider the Lie bracket*

$$[(a, X, a'), (b, Y, b')]_{\mathfrak{g}} = (\sigma(X, Y), a'D_{\mathfrak{s}}Y - b'D_{\mathfrak{s}}X + [X, Y]_{\mathfrak{s}}, 0) \quad (22)$$

and the 1-forms $\omega, \eta \in \mathfrak{g}^*$ given by

$$\omega(a, X, a') = a', \quad \eta(a, X, a') = a \quad (23)$$

for $(a, X, a'), (b, Y, b') \in \mathfrak{g}$. Then (ω, η) is a lcs structure of the first kind on the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ with central Lee vector $V = (1, 0, 0) \in \mathbb{R} \oplus \mathfrak{s} \oplus \mathbb{R}$ and anti-Lee vector $U = (0, 0, 1) \in \mathbb{R} \oplus \mathfrak{s} \oplus \mathbb{R}$.

Proof. Let (\mathfrak{s}, σ) be a symplectic Lie algebra, $D_{\mathfrak{s}}: \mathfrak{s} \rightarrow \mathfrak{s}$ a symplectic derivation and $[\cdot, \cdot]_{\mathfrak{g}}$ (resp. ω and η) the bracket (resp. the 1-forms) on \mathfrak{g} defined by (22) (resp. by (23)). Then, using Proposition 5.8 and Theorem 5.9, we deduce that $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ is a lcs Lie algebra of the first kind with lcs structure (ω, η) .

Conversely, suppose that (ω, η) is a lcs structure of the first kind on a Lie algebra \mathfrak{g} of dimension $2n + 2$ and that the Lee vector V belongs to $\mathcal{Z}(\mathfrak{g})$. Denote by (\mathfrak{h}, θ) the contact Lie subalgebra associated with \mathfrak{g} and by D the corresponding contact derivation. Then, V is the Reeb vector R of \mathfrak{h} and, therefore, $R \in \mathcal{Z}(\mathfrak{h})$. This implies that the quotient vector space $\mathfrak{s} = \mathfrak{h}/\langle R \rangle$ is a symplectic Lie algebra with symplectic structure σ and that \mathfrak{h} is the central extension of \mathfrak{s} by σ . Furthermore, using Proposition 5.8, we have that D may be given in terms of a symplectic derivation $D_{\mathfrak{s}}$ on \mathfrak{s} and, in addition, the Lie algebra structure and the lcs structure on \mathfrak{g} are given by (22) and (23), respectively. \square

Now, we may introduce the following definition.

Definition 5.12. The vector space $\mathfrak{g} = \mathbb{R} \oplus \mathfrak{s} \oplus \mathbb{R}$ in Theorem 5.11 endowed with the Lie algebra structure (22) and the lcs structure of the first kind (23) is called the *lcs extension* of the symplectic Lie algebra (\mathfrak{s}, σ) by the derivation $D_{\mathfrak{s}}$.

Remark 5.13. With the notation introduced in Sections 2.3 and 2.4 the lcs extension of (\mathfrak{s}, σ) by $D_{\mathfrak{s}}$ can be denoted $(\mathbb{R} \odot_{\sigma} \mathfrak{s}) \rtimes_D \mathbb{R}$, where D is the contact derivation of $\mathbb{R} \odot_{\sigma} \mathfrak{s}$ determined by $D_{\mathfrak{s}}$.

In some cases, the symplectic Lie algebra in Theorem 5.11 may in turn be obtained as a symplectic double extension of another symplectic Lie algebra whose dimension is $\dim \mathfrak{s} - 2$.

We recall the construction of the double extension (\mathfrak{s}, σ) of a symplectic Lie algebra $(\mathfrak{s}_1, \sigma_1)$ by a derivation $D_{\mathfrak{s}_1}$ and an element $Z_1 \in \mathfrak{s}_1$ (for more details, see [18, 51]).

It is clear that the 2-form $D_{\mathfrak{s}_1}^* \sigma_1$ on \mathfrak{s}_1 given by

$$(D_{\mathfrak{s}_1}^* \sigma_1)(X_1, Y_1) = \sigma_1(D_{\mathfrak{s}_1} X_1, Y_1) + \sigma_1(X_1, D_{\mathfrak{s}_1} Y_1), \quad (24)$$

is a 2-cocycle. So, we can consider the central extension $\mathfrak{h}_1 = \mathbb{R} \odot_{D_{\mathfrak{s}_1}^* \sigma_1} \mathfrak{s}_1$ with bracket given by (7).

Now let $(-i_{Z_1} \sigma_1, -D_{\mathfrak{s}_1}): \mathfrak{h}_1 \rightarrow \mathfrak{h}_1$ be the linear map given by

$$(-i_{Z_1} \sigma_1, -D_{\mathfrak{s}_1})(a, X_1) = (-\sigma_1(Z_1, X_1), -D_{\mathfrak{s}_1} X_1);$$

then $(-i_{Z_1} \sigma_1, -D_{\mathfrak{s}_1})$ is a derivation of \mathfrak{h}_1 if and only if

$$d(i_{Z_1} \sigma_1) = -(D_{\mathfrak{s}_1}^*)^2 \sigma_1. \quad (25)$$

Assuming that $(-i_{Z_1} \sigma_1, -D_{\mathfrak{s}_1})$ is a derivation, we can consider the vector space $\mathfrak{s} = \mathfrak{h}_1 \oplus \mathbb{R}$ with the Lie algebra structure

$$\begin{aligned} [(a_1, X_1, a'_1), (b_1, Y_1, b'_1)]_{\mathfrak{s}} = & ((D_{\mathfrak{s}_1}^* \sigma_1)(X_1, Y_1) - a'_1 \sigma_1(Z_1, Y_1) + b'_1 \sigma_1(Z_1, X_1), \\ & - a'_1 D_{\mathfrak{s}_1} Y_1 + b'_1 D_{\mathfrak{s}_1} X_1 + [X_1, Y_1]_{\mathfrak{s}_1}, 0), \end{aligned}$$

that is, \mathfrak{s} is the semidirect product $\mathfrak{h}_1 \rtimes_{(-i_{Z_1} \sigma_1, -D_{\mathfrak{s}_1})} \mathbb{R}$. Moreover, the 2-form $\sigma: \mathfrak{s} \times \mathfrak{s} \rightarrow \mathbb{R}$ on \mathfrak{s} defined by

$$\sigma((a_1, X_1, a'_1), (b_1, Y_1, b'_1)) = a_1 b'_1 - a'_1 b_1 + \sigma_1(X_1, Y_1)$$

for $(a_1, X_1, a'_1), (b_1, Y_1, b'_1) \in \mathfrak{s}$, is a symplectic structure on \mathfrak{s} . The symplectic Lie algebra (\mathfrak{s}, σ) is the *double extension* of $(\mathfrak{s}_1, \sigma_1)$ by $D_{\mathfrak{s}_1}$ and Z_1 (see [18, 51]).

Now, in the presence of a symplectic derivation $D_{\mathfrak{s}}$ of (\mathfrak{s}, σ) , we can use Theorem 5.11. In fact, we have the following result

Corollary 5.14. *Assume that we have the following data:*

1. *a symplectic Lie algebra $(\mathfrak{s}_1, \sigma_1)$ of dimension $2n - 2$;*
2. *a derivation $D_{\mathfrak{s}_1}$ of \mathfrak{s}_1 and an element $Z_1 \in \mathfrak{s}_1$ such that (25) holds and*
3. *a symplectic derivation $D_{\mathfrak{s}}$ of the symplectic double extension (\mathfrak{s}, σ) of $(\mathfrak{s}_1, \sigma_1)$ by $D_{\mathfrak{s}_1}$ and Z_1 .*

Then the vector space $\mathfrak{g} = (\mathbb{R} \oplus \mathfrak{s}) \oplus \mathbb{R}$ endowed with the Lie algebra structure given by (22) and the lcs structure of the first kind given by (23) is a lcs Lie algebra of the first kind of dimension $2n + 2$ with central Lee vector.

5.5 Lcs structures on nilpotent Lie algebras

In this section we focus on lcs structures on nilpotent Lie algebras. Our first result is:

Theorem 5.15. *Let \mathfrak{s} be a nilpotent Lie algebra endowed with a symplectic structure and let \mathfrak{g} be a nilpotent Lie algebra endowed with a lcs structure whose Lee form is non-zero.*

1. *The lcs extension of \mathfrak{s} by a symplectic nilpotent derivation is a nilpotent Lie algebra endowed with a lcs structure of the first kind with central Lee vector.*
2. *The lcs structure on \mathfrak{g} is of the first kind, the Lee vector is central and \mathfrak{g} is the lcs extension of a symplectic nilpotent Lie algebra by a symplectic nilpotent derivation.*

Proof. To 1. The central extension of a nilpotent Lie algebra by a 2-cocycle is again nilpotent. Thus, if (\mathfrak{s}, σ) is a symplectic nilpotent Lie algebra, $\mathfrak{h} = \mathbb{R} \odot_{\sigma} \mathfrak{s}$ is a contact nilpotent Lie algebra. Moreover, if $D_{\mathfrak{s}}$ is a symplectic nilpotent derivation of \mathfrak{s} and D is the corresponding contact derivation of \mathfrak{h} then, from (20), it follows that D is nilpotent. Therefore, $\mathfrak{h} \rtimes_D \mathbb{R}$ (that is, the lcs extension of \mathfrak{s} by $D_{\mathfrak{s}}$), is also a nilpotent Lie algebra. Finally, from Theorem 5.11, we have that the Lee vector of the lcs structure on \mathfrak{g} is central.

To 2. From Corollary 2.8 and Proposition 5.5, we deduce that the lcs structure on \mathfrak{g} is of the first kind. Now, let V be the Lee vector of \mathfrak{g} . By Theorem 5.9, the Lie subalgebra $\mathfrak{h} = \ker(\omega)$ is a contact Lie algebra and V is just the Reeb vector R of \mathfrak{h} . In addition, since \mathfrak{g} is nilpotent, so is \mathfrak{h} and we have that $\mathcal{Z}(\mathfrak{h}) = \langle R \rangle$ (see [19, 20]). This, together with (18), implies that $V \in \mathcal{Z}(\mathfrak{g})$.

Thus, from Theorem 5.11, it follows that \mathfrak{g} is the lcs extension of a symplectic Lie algebra \mathfrak{s} by symplectic derivation $D_{\mathfrak{s}}$ on \mathfrak{s} . In fact, using Theorems 5.9 and 5.11, we deduce that \mathfrak{s} is the quotient Lie algebra $\mathfrak{h}/\langle R \rangle \simeq \mathfrak{g}/\langle U, V \rangle$, with U the anti-Lee vector of \mathfrak{g} , and $D_{\mathfrak{s}}$ is the derivation on \mathfrak{s} induced by the operator $\text{ad}_U : \mathfrak{g} \rightarrow \mathfrak{g}$. So, \mathfrak{s} is nilpotent and, using that the endomorphism ad_U is nilpotent, we conclude that the derivation $D_{\mathfrak{s}}$ also is nilpotent. \square

On the other hand, one may prove that the symplectic double extension of a nilpotent Lie algebra by a nilpotent derivation is a symplectic nilpotent Lie algebra. In fact, in [51] (see also [18]), the authors prove that every symplectic nilpotent Lie algebra of dimension $2n$ may be obtained by a sequence of $n - 1$ symplectic double extensions by nilpotent derivations from the abelian Lie algebra of dimension 2. Hence, using these facts and Theorem 5.15, we deduce the following result

Theorem 5.16. *1. Under the same hypotheses as in Corollary 5.14 if, in addition, the derivations $D_{\mathfrak{s}_1}$ and $D_{\mathfrak{s}}$ on the symplectic nilpotent Lie algebras \mathfrak{s}_1 and \mathfrak{s} , respectively, are nilpotent then the lcs Lie algebra $(\mathbb{R} \oplus \mathfrak{s}) \oplus \mathbb{R}$ is also nilpotent.*

2. *Every lcs nilpotent Lie algebra of dimension $2n + 2$ with non-zero Lee 1-form may be obtained as the lcs extension of a $2n$ -dimensional symplectic nilpotent Lie algebra \mathfrak{s} by a symplectic nilpotent derivation and, in turn, the symplectic nilpotent Lie algebra \mathfrak{s} may be obtained by a*

sequence of $n - 1$ symplectic double extensions by nilpotent derivations from the abelian Lie algebra of dimension 2.

For the next result, we recall the notion of characteristic filtration of a nilpotent Lie algebra (see [12, 61]). Let \mathfrak{g} be a nilpotent Lie algebra and let $(\Lambda^\bullet \mathfrak{g}^*, d)$ be the associated Chevalley-Eilenberg complex. Consider the following subspaces of \mathfrak{g}^* :

$$W_1 = \ker(d), \quad W_k = d^{-1}(\wedge^2 W_{k-1}), \quad k \geq 2. \quad (26)$$

It is immediate to see that $W_{k-1} \subset W_k$, hence $\{W_k\}_k$ is a filtration of \mathfrak{g}^* , intrinsically defined. The nilpotency of \mathfrak{g} implies that there exists $m \in \mathbb{N}$ such that $W_m = \mathfrak{g}^*$. If $W_m = \mathfrak{g}^*$ but $W_{m-1} \neq \mathfrak{g}^*$, one says that \mathfrak{g} is m -step nilpotent. In particular, 1-step nilpotent Lie algebras are abelian.

Definition 5.17. Let \mathfrak{g} be a nilpotent Lie algebra. The filtration $\{W_k\}_k$ of \mathfrak{g}^* , defined by (26), is the *characteristic filtration* of \mathfrak{g}^* .

Let \mathfrak{g} be a nilpotent Lie algebra and let $\{W_k\}_k$ be the characteristic filtration of \mathfrak{g}^* . Define

$$F_1 = W_1, \quad F_k = W_k / W_{k-1}, \quad k \geq 2.$$

Clearly one has $\mathfrak{g}^* = \oplus_k F_k$, but the splitting is not canonical. Nevertheless, the numbers $f_k = \dim(F_k)$ are invariants of \mathfrak{g}^* .

Proposition 5.18. Let \mathfrak{g} be an m -step nilpotent Lie algebra of dimension $2n$. Assume \mathfrak{g} is endowed with a lcs structure. Then $f_m = 1$.

Proof. By Theorem 5.15, the lcs structure is of the first kind; we denote it (ω, η) . It is sufficient to prove that, if $f_m \geq 2$, the Lie algebra \mathfrak{g} can not admit any lcs structure of the first kind. Let $\mathcal{B} = \langle e_1^1, \dots, e_{2n-i}^{m-1}, e_{2n-i+1}^m, \dots, e_{2n-1}^m, e_{2n}^m \rangle$ be a basis of \mathfrak{g}^* adapted to the filtration $\{W_k\}$. By definition, this means that the collection $\{e_j^k\}_j$ spans F_k . Assume that $f_m \geq 2$; hence, $i \geq 2$. Set for convenience $y = e_{2n-1}^m$ and $z = e_{2n}^m$. Since $d\omega = 0$, by making a suitable change of variables in F_1 , we can assume that ω is one of the generators of F_1 , indeed we can take $\omega = e_1^1$. Let $\mathfrak{h} = \ker(\omega)$; then \mathfrak{h} is a nilpotent Lie algebra, the characteristic filtration of \mathfrak{h}^* is $\widetilde{W}_1 = W_1 / \langle \omega \rangle$, $\widetilde{W}_k = W_k$, $k \geq 2$ and $\widetilde{\mathcal{B}} = \langle e_2^1, \dots, e_{2n-i}^{m-1}, e_{2n-i+1}^m, \dots, e_{2n-2}^m, y, z \rangle$ is a basis of \mathfrak{h}^* adapted to $\{\widetilde{W}_k\}$. By Theorem 5.9, \mathfrak{h} is a contact Lie algebra, with contact form θ given by the restriction of η to \mathfrak{h} . In particular the rank of $d\theta$ is maximal on $\ker(\theta)$. This means that if $X \in \ker(\theta)$, $\iota_X d\theta \neq 0$. Expand θ on the basis $\widetilde{\mathcal{B}}$,

$$\theta = a_2 e_2^1 + \dots + \dots + a_{2n-2} e_{2n-2}^m + by + cz, \quad a_j, b, c \in \mathbb{R}.$$

Let $\langle X_2, \dots, X_{2n-2}, Y, Z \rangle$ be the basis of \mathfrak{h} dual to $\widetilde{\mathcal{B}}$. The vector $T = cY - bZ$ clearly belongs to $\ker(\theta)$. By hypothesis, $dy, dz \in \Lambda^2 \widetilde{W}_{m-1}$, and $T \in \ker(e_\ell^k) \forall k, \ell$, hence $\iota_T d\theta = 0$. Thus $\theta \wedge (d\theta)^{n-1}$ has a kernel, contradicting the fact that (\mathfrak{h}, θ) is a contact Lie algebra. \square

To conclude this section, we use the classification of nilpotent Lie algebras in dimension 4 and 6 to determine which of them carry a locally conformal symplectic structure. We refer to Section 2.6 for the notation.

Proposition 5.19. Suppose $(\mathfrak{g}, \omega, \eta)$ is a nilpotent Lie algebra of dimension 4 endowed with a lcs structure. Then \mathfrak{g} is isomorphic to one of the following Lie algebras:

- $\mathfrak{g}_1 = (0, 0, 0, 12)$, $\omega = e^3$, $\eta = e^4$;
- $\mathfrak{g}_2 = (0, 0, 12, 13)$, $\omega = e^2$, $\eta = e^4$.

Proof. There are three isomorphism classes of nilpotent Lie algebras in dimension 4 (see [12] for instance): the two listed above and the abelian one, which is clearly not lcs. \square

Remark 5.20. Both Lie algebras in Proposition 5.19 are symplectic. \mathfrak{g}_1 admits a complex structure and is the Lie algebra of the so called *Kodaira-Thurston* nilmanifold, that was mentioned in the introduction. \mathfrak{g}_2 does not admit any complex structure (see [61]) and is the Lie algebra of the 4-dimensional nilpotent Lie group G considered in Section 3.1.

Proposition 5.21. *Suppose $(\mathfrak{g}, \omega, \eta)$ is a nilpotent Lie algebra of dimension 6 endowed with a lcs structure. Then \mathfrak{g} is isomorphic to one of the Lie algebras contained in Table 2.*

The third and fourth column in Table 2 contain two labellings of nilpotent Lie algebras, according to [12] and [17] respectively. We also compare lcs structures with other structures on 6-dimensional nilpotent Lie algebras, namely symplectic and complex structures.

Table 2: Locally conformal symplectic nilpotent Lie algebras in dimension 6

Lie algebra	ω	η	[12]	[17]	Symplectic	Complex
$(0, 0, 0, 0, 0, 12 + 34)$	e^5	e^6	$L_{5,1} \oplus A_1$	\mathfrak{h}_3	\times	\checkmark
$(0, 0, 0, 0, 12, 15 + 34)$	e^2	e^6	$L_{6,3}$	\mathfrak{h}_{20}	\times	\times
$(0, 0, 0, 0, 12, 15 + 23)$	e^4	e^6	$L_{5,3} \oplus A_1$	\mathfrak{h}_9	\checkmark	\checkmark
$(0, 0, 0, 12, 13, 15 + 24)$	e^3	e^6	$L_{6,7}$	\mathfrak{h}_{18}	\times	\times
$(0, 0, 0, 12, 13, 24 + 35)$	e^1	e^6	$L_{6,8}^+$	\mathfrak{h}_{19}^-	\times	\checkmark
$(0, 0, 0, 12, 13, 24 - 35)$	e^1	e^6	$L_{6,8}^-$	\mathfrak{h}_{19}^+	\times	\times
$(0, 0, 0, 12, 14, 15 + 24)$	e^3	e^6	$L_{5,6} \oplus A_1$	\mathfrak{h}_{22}	\checkmark	\times
$(0, 0, 0, 12, 14, 15 + 23 + 24)$	e^3	e^6	$L_{6,14}$	\mathfrak{h}_{24}	\checkmark	\times
$(0, 0, 0, 12, 14 + 23, 15 - 34)$	e^2	e^6	$L_{6,15}$	\mathfrak{h}_{27}	\checkmark	\times
$(0, 0, 12, 13, 14, 25 - 34)$	e^1	e^6	$L_{6,20}$	\mathfrak{h}_{31}	\times	\times
$(0, 0, 12, 13, 14 + 23, 25 - 34)$	e^1	e^6	$L_{6,22}$	\mathfrak{h}_{32}	\times	\times

Proof. By Proposition 5.18, we can restrict to those Lie algebras whose characteristic filtration ends with a one-dimensional space. It is then enough to show that if \mathfrak{g} is one of the 6-dimensional nilpotent Lie algebras which does not appear in the above list, there exists no contact ideal of \mathfrak{g} and then apply Theorem 5.9.

As an example, consider the Lie algebra $L_{6,13} = (0, 0, 0, 12, 14, 15 + 23)$ (in the notation of [12]). Then $L_{6,13}$ has a basis $\mathcal{B} = \langle e^1, \dots, e^6 \rangle$ whose closed elements are e^1, e^2 and e^3 . The generic closed element has the form $\omega = a_1 e^1 + a_2 e^2 + a_3 e^3$, hence $\mathfrak{h} = \ker(\omega)$ has a basis

$$\langle a_2 e_1 - a_1 e_2, a_3 e_1 - a_1 e_3, e_4, e_5, e_6 \rangle =: \langle f_1, f_2, f_3, f_4, f_5 \rangle,$$

where $\langle e_1, \dots, e_6 \rangle$ is the basis dual to \mathcal{B} . A computation shows that, with respect to the basis $\langle f_1, f_2, f_3, f_4, f_5 \rangle$ basis and its dual basis, the Chevalley-Eilenberg differential in \mathfrak{h}^* is given by

$$df^1 = 0 = df^2, \quad df^3 = a_1 a_3 f^{12}, \quad df^4 = a_2 f^{13} + a_3 f^{23} \quad \text{and} \quad df^5 = a_1 a_2 f^{12} + a_2 f^{14} + a_3 f^{24}.$$

We proceed case by case:

- if $a_1 = 0$ or $a_3 = 0$, a change of basis shows that \mathfrak{h} is isomorphic to $(0, 0, 0, 12, 14)$, which is not a contact algebra;
- if $a_2 = 0$, a change of basis shows that \mathfrak{h} is isomorphic to $(0, 0, 12, 13, 14)$, which is not a contact algebra;
- if $a_1 a_2 a_3 \neq 0$, a change of basis shows that \mathfrak{h} is isomorphic to $(0, 0, 12, 13, 14)$ as well.

We conclude that $L_{6,13}$ has no suitable contact ideal and, by Theorem 5.9, it is not locally conformal symplectic. \square

According to [61], the five nilpotent Lie algebras in Table 2 that admit neither a symplectic nor a complex structure are the only such examples in dimension 6. Since all of them admit a lcs structure, we obtain the following result:

Corollary 5.22. *A 6-dimensional nilpotent Lie algebra carries at least one among symplectic, complex and locally conformal symplectic structures.*

6 Locally conformal symplectic Lie groups

First of all, we will introduce, in a natural way, the notion of a locally conformal symplectic (lcs) Lie group.

Definition 6.1. A Lie group G of dimension $2n$ ($n \geq 2$) is said to be a *locally conformal symplectic (lcs) Lie group* if it admits a closed left-invariant 1-form $\overleftarrow{\omega}$ and a non-degenerate left-invariant 2-form $\overleftarrow{\Phi}$ such that

$$d\overleftarrow{\Phi} = \overleftarrow{\omega} \wedge \overleftarrow{\Phi}.$$

The lcs structure is said to be *of the first kind* if G admits a left-invariant 1-form $\overleftarrow{\eta}$ such that $\Phi = d\overleftarrow{\eta} + \overleftarrow{\eta} \wedge \overleftarrow{\omega}$ and $\overleftarrow{\omega} \wedge \overleftarrow{\eta} \wedge (d\overleftarrow{\eta})^{n-1}$ is a volume form on G .

If e is the identity element of a Lie group G and $\mathfrak{g} = T_e G$ is the Lie algebra of G then it is clear that G is a lcs Lie group (resp. lcs Lie group of the first kind) if and only if \mathfrak{g} is a lcs Lie algebra (resp. lcs Lie algebra of the first kind). In fact, if $(\overleftarrow{\omega}, \overleftarrow{\Phi})$ is a lcs structure on G then (ω, Φ) is a lcs structure on \mathfrak{g} , with

$$\omega = \overleftarrow{\omega}(e), \quad \Phi = \overleftarrow{\Phi}(e).$$

In a similar way, if $(\overleftarrow{\omega}, \overleftarrow{\eta})$ is a lcs structure of the first kind on G then (ω, η) is a lcs structure of the first kind on \mathfrak{g} , with

$$\omega = \overleftarrow{\omega}(e), \quad \eta = \overleftarrow{\eta}(e).$$

Next, we will restrict our attention to lcs Lie groups of the first kind. More precisely, we will discuss the relation between lcs Lie groups of the first kind and contact (resp. symplectic) Lie groups.

6.1 Relation with contact Lie groups

It is well-known that a Lie group H of dimension $2n - 1$ is a contact Lie group if it admits a contact left-invariant 1-form $\overleftarrow{\eta}$, that is, $\overleftarrow{\eta} \wedge (d\overleftarrow{\eta})^{n-1}$ is a volume form on H (see, for instance, [19, 20]).

As in the lcs case, a Lie group H with Lie algebra \mathfrak{h} is a contact Lie group if and only if \mathfrak{h} is a contact Lie algebra. In fact, if $\overleftarrow{\eta}$ is a left-invariant contact 1-form on H then $\eta = \overleftarrow{\eta}(e)$ is a contact structure on \mathfrak{h} .

Next, using contact Lie groups, we will present the typical example of a lcs Lie group of the first kind.

Example 6.2. Let $(H, \overleftarrow{\eta})$ be a contact Lie group of dimension $2n - 1$ and $\phi: \mathbb{R} \rightarrow \text{Aut}(H)$ the flow of a contact multiplicative vector field \mathcal{M} on H . In other words,

$$\phi: \mathbb{R} \rightarrow \text{Aut}(H)$$

is a representation of the abelian Lie group \mathbb{R} on H and

$$\phi_t^*(\overleftarrow{\eta}) = \overleftarrow{\eta}, \quad \text{for } t \in \mathbb{R}. \quad (27)$$

Then, we can consider the semidirect product $G = H \rtimes_{\phi} \mathbb{R}$ whose Lie algebra \mathfrak{g} is the semidirect product $\mathfrak{g} = \mathfrak{h} \rtimes_D \mathbb{R}$, with $D: \mathfrak{h} \rightarrow \mathfrak{h}$ the derivation in \mathfrak{h} induced by the representation ϕ (see Section 2.3). Denote by η the contact structure on the Lie algebra \mathfrak{h} , that is, $\eta = \overleftarrow{\eta}(\mathbf{e})$, \mathbf{e} being the identity element in H . Then, from (27), it follows that $\mathcal{L}_{\mathcal{M}}\overleftarrow{\eta} = 0$, which implies that

$$D^*\eta = 0,$$

that is, D is a contact derivation. Thus, if $\omega = (0, 1) \in \mathfrak{h}^* \oplus \mathbb{R} = \mathfrak{g}^*$ then, using Theorem 5.9, we deduce that the couple (ω, η) is a lcs structure of the first kind on \mathfrak{g} . Therefore, $(\overleftarrow{\omega}, \overleftarrow{\eta})$ is a lcs structure of the first kind on $G = H \rtimes_{\phi} \mathbb{R}$.

Remark 6.3. Note that if \overleftarrow{R} is the Reeb vector field of H , with $R \in \mathfrak{h}$, then $(\overleftarrow{R}, 0)$ is just the Lee vector field of $G = H \rtimes_{\phi} \mathbb{R}$.

The previous example is the model of a connected simply connected lcs Lie group of the first kind. More precisely, two lcs Lie groups of the first kind (G, ω, η) and (G', ω', η') are isomorphic if there exists a Lie group isomorphism $\Psi: G \rightarrow G'$ such that

$$\Psi^*\omega' = \omega \quad \text{and} \quad \Psi^*\eta' = \eta.$$

Then, one may prove the following result

Theorem 6.4. *Let G be a connected simply connected lcs Lie group of the first kind. If \mathfrak{g} is the Lie algebra of G then \mathfrak{g} is isomorphic to a semidirect product of a contact Lie algebra \mathfrak{h} with \mathbb{R} . Moreover, if H is a connected simply connected Lie group with Lie algebra \mathfrak{h} , then G is isomorphic to a semidirect product $H \rtimes_{\phi} \mathbb{R}$, where $\phi: \mathbb{R} \rightarrow \text{Aut}(H)$ is a contact representation of the abelian Lie group \mathbb{R} on H and the lcs structure on $H \rtimes_{\phi} \mathbb{R}$ is given as in Example 6.2.*

Proof. Let \mathfrak{g} be the Lie algebra of G and (ω, η) the lcs structure of the first kind on \mathfrak{g} . Then, using Theorem 5.9, we deduce the following facts:

- η induces a contact structure on the ideal $\mathfrak{h} = \ker(\omega) \subset \mathfrak{g}$;
- there exists a contact derivation $D: \mathfrak{h} \rightarrow \mathfrak{h}$ such that \mathfrak{g} is isomorphic to the semidirect product $\mathfrak{h} \rtimes_D \mathbb{R}$ and
- under the previous isomorphism ω is the 1-form $(0, 1)$ in $\mathfrak{h}^* \oplus \mathbb{R}$.

Now, let H be a connected simply connected Lie group with Lie algebra \mathfrak{h} . Then, H is a contact Lie group. In addition, the derivation $D: \mathfrak{h} \rightarrow \mathfrak{h}$ induces a representation

$$\phi: \mathbb{R} \rightarrow \text{Aut}(H).$$

In fact, if \mathcal{M} is the multiplicative vector field on H whose flow is ϕ then, using (4) and the fact that D is a contact derivation, we deduce that

$$\mathcal{L}_{\mathcal{M}}\overleftarrow{\eta} = 0,$$

which implies that ϕ is a contact representation, that is,

$$\phi_t^* \overleftarrow{\eta} = \overleftarrow{\eta}, \quad \text{for } t \in \mathbb{R}.$$

On the other hand, it is clear that the Lie algebra of the connected simply connected Lie group $H \rtimes_{\phi} \mathbb{R}$ is $\mathfrak{h} \rtimes_D \mathbb{R} \simeq \mathfrak{g}$. This ends the proof of the result. \square

Remark 6.5. Let Γ be a lattice in the contact Lie group H and let $r\mathbb{Z}$ be an integer lattice of \mathbb{R} such that the restriction to $r\mathbb{Z}$ of the representation ϕ (in Theorem 6.4) takes values in $\text{Aut}(\Gamma)$. Then $\Gamma \rtimes_{\phi} (r\mathbb{Z})$ is a lattice in $H \rtimes_{\phi} \mathbb{R}$ and $M = (\Gamma \rtimes_{\phi} (r\mathbb{Z})) \backslash (H \rtimes_{\phi} \mathbb{R})$ is a compact manifold. Furthermore, since the lcs structure of the first kind on $H \rtimes_{\phi} \mathbb{R}$ is left-invariant, it induces a lcs structure of the first kind on M . Thus, M is a compact lcs manifold of the first kind. This construction can also be described as a mapping torus. More precisely, notice that the compact quotient $N := \Gamma \backslash H$ carries a contact structure; moreover, $\bar{\phi} := \phi(r)$ gives a strict contactomorphism $\bar{\phi}: H \rightarrow H$ which preserves Γ , hence descends to a strict contactomorphism of N , denoted again $\bar{\phi}$. We can form its mapping torus

$$N_{\bar{\phi}} = N \times_{(\bar{\phi}, r)} \mathbb{R}.$$

The map $M \rightarrow N_{\bar{\phi}}$ given by $[(h, t)] \mapsto [([h], t)]$ is a diffeomorphism.

6.2 Relation with symplectic Lie groups

A Lie group S of dimension $2n$ is said to be a symplectic Lie group if it admits a left-invariant 2-form $\overleftarrow{\sigma}$ which is closed and non-degenerate (see, for instance, [9, 18, 43]).

Let S be a Lie group with identity e and let $\mathfrak{s} = T_e S$ be its Lie algebra; it is clear that S is a symplectic Lie group if and only if \mathfrak{s} is a symplectic Lie algebra. In fact, if $\overleftarrow{\sigma}$ is a left-invariant symplectic structure on S then $\sigma = \overleftarrow{\sigma}(e) \in \Lambda^2 \mathfrak{s}^*$ is a symplectic structure on \mathfrak{s} .

Next, we will discuss the relation between symplectic Lie groups and a particular class of lcs Lie groups of the first kind.

6.2.1 Lcs Lie groups of the first kind with bi-invariant Lee vector field

First of all, we present the following example.

Example 6.6. Let S be a Lie group of dimension $2n$ with Lie algebra \mathfrak{s} . Moreover, suppose given:

- a 2-cocycle $\varphi: S \times S \rightarrow \mathbb{R}$ on S with values in \mathbb{R} such that the corresponding 2-cocycle $\sigma: \mathfrak{s} \times \mathfrak{s} \rightarrow \mathbb{R}$ on \mathfrak{s} with values in \mathbb{R} given by (6) is non-degenerate;
- a representation $\phi: \mathbb{R} \rightarrow \text{Aut}(S)$ of the abelian Lie group \mathbb{R} on S such that

$$\phi_u^* \overleftarrow{\sigma} = \overleftarrow{\sigma}, \quad \text{for every } u \in \mathbb{R}.$$

Note that the first condition implies that (\mathfrak{s}, σ) is a symplectic Lie algebra, hence S is a symplectic Lie group with symplectic form $\overleftarrow{\sigma}$.

Therefore, the central extension $\mathbb{R} \odot_{\sigma} \mathfrak{s}$ is a contact Lie algebra with contact structure $\eta = (1, 0) \in \mathbb{R} \oplus \mathfrak{s}^*$ and central Reeb vector $R = (1, 0) \in \mathbb{R} \odot_{\sigma} \mathfrak{s}$. This implies that the central extension $\mathbb{R} \odot_{\varphi} S$ is a contact Lie group with contact structure $\overleftarrow{\eta}$ and bi-invariant Reeb vector field \overleftarrow{R} . Note that the left-invariant vector field \overleftarrow{R} is bi-invariant since R belongs to the center of $\mathbb{R} \odot_{\sigma} \mathfrak{s}$.

Now, denote by $D_{\mathfrak{s}}: \mathfrak{s} \rightarrow \mathfrak{s}$ the symplectic derivation induced by ϕ . Then, from Proposition 5.8, it follows that the linear map $D: \mathbb{R} \odot_{\sigma} \mathfrak{s} \rightarrow \mathbb{R} \odot_{\sigma} \mathfrak{s}$ given by

$$D(u, X) = (0, D_{\mathfrak{s}} X), \quad \text{for } u \in \mathbb{R} \text{ and } X \in \mathfrak{s} \quad (28)$$

is a contact derivation. Thus, it induces a contact representation $\tilde{\phi}: \mathbb{R} \rightarrow \text{Aut}(\mathbb{R} \odot_{\varphi} S)$ and the semidirect product $G = (\mathbb{R} \odot_{\varphi} S) \rtimes_{\tilde{\phi}} \mathbb{R}$ is a lcs Lie group of the first kind with bi-invariant Lee vector field $(\overline{R}, 0)$ (see Example 6.2).

We remark the following facts:

- \mathfrak{g} , the Lie algebra of G , is the lcs extension $(\mathbb{R} \odot_{\sigma} \mathfrak{s}) \rtimes_D \mathbb{R}$ of the symplectic Lie algebra (\mathfrak{s}, σ) by the derivation $D_{\mathfrak{s}}$ (compare with Remark 5.13);
- the element $(R, 0) \in \mathfrak{g}$ is central; this implies that the vector field $(\overline{R}, 0)$ on G is bi-invariant.

Next, we will present a more explicit description of the contact representation $\tilde{\phi}: \mathbb{R} \rightarrow \text{Aut}(\mathbb{R} \odot_{\varphi} S)$ in terms of the symplectic representation $\phi: \mathbb{R} \rightarrow \text{Aut}(S)$.

As we know, the multiplication in $\mathbb{R} \odot_{\varphi} S$ is given by

$$(u, s)(u', s') = (u + u' + \varphi(s, s'), ss'), \quad \text{for } (u, s), (u', s') \in \mathbb{R} \odot_{\varphi} S. \quad (29)$$

Thus, we have the following expressions of the left-invariant vector fields on $\mathbb{R} \odot_{\varphi} S$,

$$\overline{(1, 0)} = \frac{\partial}{\partial u}, \quad \overline{(0, X)}(u, s) = \frac{d}{dr} \Big|_{r=0} \varphi(s, \exp(rX)) \frac{\partial}{\partial u} \Big|_{(u, s)} + \overline{X}(s), \quad (30)$$

for $X \in \mathfrak{s}$ and $(u, s) \in \mathbb{R} \odot_{\varphi} S$.

Now, let $\widetilde{\mathcal{M}}$ be the multiplicative vector field on $\mathbb{R} \odot_{\varphi} S$ whose flow is $\tilde{\phi}$,

$$\widetilde{\mathcal{M}}(u, s) = \tilde{\psi}(u, s) \frac{\partial}{\partial u} \Big|_{(u, s)} + \overline{\mathcal{M}}(u, s), \quad \text{for } (u, s) \in \mathbb{R} \odot_{\varphi} S,$$

with $\tilde{\psi} \in \mathcal{C}^{\infty}(\mathbb{R} \odot_{\varphi} S)$ and $\overline{\mathcal{M}}: \mathbb{R} \odot_{\varphi} S \rightarrow TS$ a time-dependent vector field on S .

From (4), (28) and (30), we deduce that

$$0 = \overline{D(1, 0)} = \left[\frac{\partial}{\partial u}, \widetilde{\mathcal{M}} \right]$$

which implies that

$$\widetilde{\mathcal{M}}(u, s) = \tilde{\psi}(s) \frac{\partial}{\partial u} \Big|_{(u, s)} + \overline{\mathcal{M}}(s), \quad \text{for } (u, s) \in \mathbb{R} \odot_{\varphi} S,$$

with $\tilde{\psi} \in \mathcal{C}^{\infty}(S)$ and $\overline{\mathcal{M}} \in \mathfrak{X}(S)$.

So, using (29) and the fact that $\widetilde{\mathcal{M}}$ is multiplicative, we conclude that $\overline{\mathcal{M}}$ also is multiplicative. Moreover, from (28) and (30), it follows that the derivation on \mathfrak{s} associated with $\overline{\mathcal{M}}$ is just $D_{\mathfrak{s}}: \mathfrak{s} \rightarrow \mathfrak{s}$. Thus, $\overline{\mathcal{M}} = \mathcal{M}$, with \mathcal{M} the multiplicative vector field on S whose flow is ϕ , and

$$\widetilde{\mathcal{M}}(u, s) = \tilde{\psi}(s) \frac{\partial}{\partial u} \Big|_{(u, s)} + \mathcal{M}(s).$$

Therefore, the flow $\tilde{\phi}: \mathbb{R} \rightarrow \text{Aut}(\mathbb{R} \odot_{\varphi} S)$ of $\widetilde{\mathcal{M}}$ has the form

$$\tilde{\phi}_t(u, s) = (u + \tilde{\chi}_t(s), \phi_t(s))$$

and

$$\widetilde{\mathcal{M}}(u, s) = \frac{d}{dt} \Big|_{t=0} (\tilde{\chi}_t(s)) \frac{\partial}{\partial u} \Big|_{(u, s)} + \mathcal{M}(s) \quad (31)$$

with $\tilde{\chi} \in \mathcal{C}^\infty(\mathbb{R} \times S)$. Since $\tilde{\phi}$ is an action of \mathbb{R} on $\mathbb{R} \odot_\varphi S$, we deduce that $\tilde{\chi}$ satisfies the relation

$$\tilde{\chi}_{t+t'}(s) = \tilde{\chi}_t(s) + \tilde{\chi}_{t'}(\phi_t(s)), \quad \text{for } t, t' \in \mathbb{R} \text{ and } s \in S. \quad (32)$$

Moreover, using (29) and the fact that $\tilde{\chi}_t \in \text{Aut}(\mathbb{R} \odot_\varphi S)$, for every $t \in \mathbb{R}$, we obtain that

$$\tilde{\chi}_t(ss') - \tilde{\chi}_t(s) - \tilde{\chi}_t(s') = \varphi(\phi_t(s), \phi_t(s')) - \varphi(s, s'). \quad (33)$$

In addition, from (30) and (31), it follows that

$$\overline{D(0, X)} = [\overline{(0, X)}, \tilde{\mathcal{M}}](-\varphi(\mathbf{e}, \mathbf{e}), \mathbf{e}) = \frac{d}{dt} \Big|_{t=0} (d_{\mathbf{e}} \tilde{\chi}_t)(X) \frac{\partial}{\partial u} \Big|_{(-\varphi(\mathbf{e}, \mathbf{e}), \mathbf{e})} + D_{\mathbf{s}} X,$$

which, using (28), implies that

$$\frac{d}{dt} \Big|_{t=0} (d_{\mathbf{e}} \tilde{\chi}_t) = 0. \quad (34)$$

Definition 6.7. The Lie group $(\mathbb{R} \odot_\varphi S) \rtimes_{\tilde{\phi}} \mathbb{R}$ endowed with the previous lcs structure of the first kind is called the *lcs extension of the Lie group S by the symplectic 2-cocycle φ and the symplectic representation $\phi: \mathbb{R} \rightarrow \text{Aut}(S)$* .

Lcs extensions of symplectic Lie groups by symplectic 2-cocycles and symplectic representations are the models of connected simply connected lcs Lie groups of the first kind with bi-invariant Lee vector field. In fact, we can prove the following result.

Theorem 6.8. *Let G be a connected simply connected lcs Lie group of the first kind with bi-invariant Lee vector field. If \mathfrak{g} is the Lie algebra of G then \mathfrak{g} is a lcs extension of a symplectic Lie algebra \mathfrak{s} and if S is a connected simply connected Lie group with Lie algebra \mathfrak{s} , we have that G is isomorphic to a lcs extension of S .*

Proof. Let \mathfrak{g} be the Lie algebra of G , let (ω, η) be the lcs structure of the first kind on \mathfrak{g} and let $V \in \mathfrak{g}$ be the Lee vector. Using that the Lee vector field \tilde{V} of G is bi-invariant, we have that V belongs to the center of \mathfrak{g} . Thus, from Theorem 5.11 and Definition 5.12, we deduce that \mathfrak{g} may be identified with the lcs extension of a symplectic Lie algebra (\mathfrak{s}, σ) by a symplectic derivation $D_{\mathbf{s}}: \mathfrak{s} \rightarrow \mathfrak{s}$. Under this identification, $\omega = ((0, 0), 1) \in (\mathbb{R} \oplus \mathfrak{s}^*) \oplus \mathbb{R}$, $\eta = ((1, 0), 0) \in (\mathbb{R} \oplus \mathfrak{s}^*) \oplus \mathbb{R}$ and $V = ((1, 0), 0) \in (\mathbb{R} \odot_\sigma \mathbb{R}) \rtimes_D \mathbb{R}$ (compare with Remark 5.13).

Now, let S be a connected simply connected Lie group with Lie algebra \mathfrak{s} and $\varphi: S \times S \rightarrow \mathbb{R}$ a 2-cocycle on S with values in \mathbb{R} such that the corresponding 2-cocycle on \mathfrak{s} with values in \mathbb{R} is just σ (see Section 2.4). Then, the central extension $H = \mathbb{R} \odot_\varphi S$ is a connected simply connected contact Lie group with Lie algebra $\mathbb{R} \odot_\sigma \mathfrak{s}$, contact structure $\tilde{\eta} = \overline{(1, 0)}$ and bi-invariant Reeb vector field $\tilde{V} = \overline{(1, 0)}$.

Next, proceeding as in the proof of Theorem 6.4, we can take a contact representation

$$\tilde{\phi}: \mathbb{R} \rightarrow \text{Aut}(\mathbb{R} \odot_\varphi S)$$

of \mathbb{R} on the contact Lie group $\mathbb{R} \odot_\varphi S$ such that the corresponding contact derivation $D: \mathbb{R} \odot_\sigma \mathfrak{s} \rightarrow \mathbb{R} \odot_\sigma \mathfrak{s}$ is given by

$$D(u, X) = (0, D_{\mathbf{s}}(X)).$$

Then, the semidirect product $(\mathbb{R} \odot_\varphi S) \rtimes_{\tilde{\phi}} \mathbb{R}$ is a connected simply connected Lie group with Lie algebra the lcs extension $(\mathbb{R} \odot_\sigma \mathfrak{s}) \rtimes_D \mathbb{R}$ of (\mathfrak{s}, σ) by $D_{\mathbf{s}}$. Thus, since \mathfrak{g} is isomorphic to $(\mathbb{R} \odot_\sigma \mathfrak{s}) \rtimes_D \mathbb{R}$, there exists a Lie group isomorphism between G and $(\mathbb{R} \odot_\varphi S) \rtimes_{\tilde{\phi}} \mathbb{R}$ and, under this isomorphism, the Lee 1-form of G is just the left-invariant 1-form $\overline{((0, 0), 1)}$, with $((0, 0), 1) \in \mathbb{R} \oplus \mathfrak{s}^* \oplus \mathbb{R}$.

Now, let $\phi: \mathbb{R} \rightarrow \text{Aut}(S)$ be a representation of the abelian Lie group \mathbb{R} on S such that the corresponding derivation on \mathfrak{s} is just $D_{\mathfrak{s}}: \mathfrak{s} \rightarrow \mathfrak{s}$. Denote by \mathcal{M} the multiplicative vector field on S whose flow is $\{\phi_t\}_{t \in \mathbb{R}}$. Then, using (4) and the fact that $D_{\mathfrak{s}}$ is a symplectic derivation, we deduce that

$$\mathcal{L}_{\mathcal{M}} \overleftarrow{\sigma} = 0,$$

which implies that ϕ is a symplectic representation, i.e.,

$$\phi_t^* \overleftarrow{\sigma} = \overleftarrow{\sigma}, \quad \text{for every } t \in \mathbb{R}.$$

Finally, proceeding as in Example 6.6, one may see that

$$\tilde{\phi}_t(u, s) = (u + \tilde{\chi}_t(s), \phi_t(s)), \quad \text{for } t, u \in \mathbb{R} \text{ and } s \in S,$$

with $\tilde{\chi}: \mathbb{R} \times S \rightarrow \mathbb{R}$ a smooth map satisfying (32), (33) and (34). \square

Remark 6.9. Let S be a connected simply connected symplectic Lie group, $\varphi: S \times S \rightarrow \mathbb{R}$ a symplectic 2-cocycle on S with values in \mathbb{R} and $\phi: \mathbb{R} \rightarrow \text{Aut}(S)$ a symplectic representation of \mathbb{R} on S . Suppose that Γ_S is a lattice in S and p, q are real numbers such that:

- the restriction of φ to $\Gamma_S \times \Gamma_S$ takes values in the integer lattice $p\mathbb{Z}$ and
- the restriction to the integer lattice $q\mathbb{Z}$ of the corresponding contact representation $\tilde{\phi}: \mathbb{R} \rightarrow \text{Aut}(\mathbb{R} \odot_{\varphi} S)$ takes values in $\text{Aut}(p\mathbb{Z} \odot_{\varphi} \Gamma_S)$.

Then, $(p\mathbb{Z} \odot_{\varphi} \Gamma_S) \rtimes_{\tilde{\phi}} q\mathbb{Z}$ is a lattice in the lcs Lie group of the first kind $(\mathbb{R} \odot_{\varphi} S) \rtimes_{\tilde{\phi}} \mathbb{R}$ and $M = ((p\mathbb{Z} \odot_{\varphi} \Gamma_S) \rtimes_{\tilde{\phi}} q\mathbb{Z}) \backslash ((\mathbb{R} \odot_{\varphi} S) \rtimes_{\tilde{\phi}} \mathbb{R})$ is a compact manifold. Furthermore, since the lcs structure of the first kind on $(\mathbb{R} \odot_{\varphi} S) \rtimes_{\tilde{\phi}} \mathbb{R}$ is left-invariant, it induces a lcs structure of the first kind on M . Thus, M is a compact lcs manifold of the first kind.

6.2.2 Symplectic extensions of symplectic Lie groups and lcs Lie groups

In some cases, the symplectic Lie group S of dimension $2n$ (in the previous section) may in turn be obtained from a symplectic Lie group S_1 of dimension $2n - 2$.

In fact, an *integrated version* of the symplectic double extension of a symplectic Lie algebra (see Section 5.3) produces S from S_1 . This process may be described as follows.

Let S_1 be a Lie group of dimension $2n - 2$ with Lie algebra \mathfrak{s}_1 and suppose that:

- there exists a 2-cocycle on S_1 with values in \mathbb{R} , $\varphi_1: S_1 \times S_1 \rightarrow \mathbb{R}$, such that the corresponding 2-cocycle on \mathfrak{s}_1 with values in \mathbb{R} , $\sigma_1: \mathfrak{s}_1 \times \mathfrak{s}_1 \rightarrow \mathbb{R}$ is non-degenerate and
- there exists a representation of the abelian group \mathbb{R} on S_1 , $\phi_1: \mathbb{R} \rightarrow \text{Aut}(S_1)$.

Under these conditions, the left-invariant 2-form $\overleftarrow{\sigma}_1$ is a symplectic structure on S_1 .

Now, let $D_{\mathfrak{s}_1}: \mathfrak{s}_1 \rightarrow \mathfrak{s}_1$ be the derivation on \mathfrak{s}_1 associated with the representation $\phi_1: \mathbb{R} \rightarrow \text{Aut}(S_1)$ and $D_{\mathfrak{s}_1}^* \sigma_1: \mathfrak{s}_1 \times \mathfrak{s}_1 \rightarrow \mathbb{R}$ the 2-cocycle on \mathfrak{s}_1 with values in \mathbb{R} given by (24). Then, we will define a 2-cocycle on S_1 with values in \mathbb{R} such that the corresponding 2-cocycle on \mathfrak{s}_1 is just $D_{\mathfrak{s}_1}^* \sigma_1$.

For this purpose, we will consider the map $(\varphi_1, \phi_1): S_1 \times S_1 \rightarrow \mathbb{R}$ given by

$$(\varphi_1, \phi_1)(s_1, s'_1) = \left. \frac{d}{dr} \right|_{r=0} \varphi_1(\phi_1(r)(s_1), \phi_1(r)(s'_1)), \quad \text{for } s_1, s'_1 \in S_1.$$

Using that $\varphi_1: S_1 \times S_1 \rightarrow \mathbb{R}$ is a 2-cocycle on S_1 and the fact that $\phi_1: \mathbb{R} \rightarrow \text{Aut}(S_1)$ is a representation, we have that (φ_1, ϕ_1) also is a 2-cocycle. In addition, if $(\sigma_1, T_{\mathbf{e}_1} \phi_1): \mathfrak{s}_1 \times \mathfrak{s}_1 \rightarrow \mathbb{R}$ is the 2-cocycle on \mathfrak{s}_1 associated with (φ_1, ϕ_1) then, from (6), it follows that

$$(\sigma_1, T_{\mathbf{e}_1} \phi_1)(X_1, X'_1) = \left. \frac{d}{dr} \right|_{r=0} \sigma_1((T_{\mathbf{e}_1} \phi_1(r))(X_1), (T_{\mathbf{e}_1} \phi_1(r))(X'_1)).$$

On the other hand, using (4), we obtain that

$$\overleftarrow{D_{\mathfrak{s}_1} X_1} = \frac{d}{dr} \Big|_{r=0} (T_{\mathfrak{e}_1} \phi_1(r))(X_1).$$

This implies that

$$(\sigma_1, T_{\mathfrak{e}_1} \phi_1)(X_1, X'_1) = \sigma_1(D_{\mathfrak{s}_1} X_1, X'_1) + \sigma_1(X_1, D_{\mathfrak{s}_1} X'_1).$$

In other words, $(\sigma_1, T_{\mathfrak{e}_1} \phi_1)$ is just the 2-cocycle $D_{\mathfrak{s}_1}^* \sigma_1$ and, thus, (φ_1, ϕ_1) is a 2-cocycle on S_1 associated with $D_{\mathfrak{s}_1}^* \sigma_1$. Therefore, we may consider the central extension $H_1 = \mathbb{R} \odot_{(\varphi_1, \phi_1)} S_1$. We remark that the multiplication in H_1 is given by

$$(r, s_1)(r', s'_1) = (r + r' + (\varphi_1, \phi_1)(s_1, s'_1), s_1 s'_1). \quad (35)$$

We will denote by \mathfrak{h}_1 the Lie algebra of H_1 . It follows that $\mathfrak{h}_1 = \mathbb{R} \odot_{D_{\mathfrak{s}_1}^* \sigma_1} \mathfrak{s}_1$ and, moreover,

$$\overleftarrow{(1, 0)} = \frac{\partial}{\partial r}, \quad \overleftarrow{(0, X_1)}(r, s_1) = \frac{d}{du} \Big|_{u=0} (\varphi_1, \phi_1)(s_1, \exp(uX_1)) \frac{\partial}{\partial r} \Big|_{(r, s_1)} + \overleftarrow{X_1}(s_1) \quad (36)$$

for $X_1 \in \mathfrak{s}_1$ and $(r, s_1) \in \mathbb{R} \odot_{(\varphi_1, \phi_1)} S_1 = H_1$.

Now, as in Section 5.4, suppose that $Z_1 \in \mathfrak{s}_1$ and that the map $(-i_{Z_1} \sigma_1, -D_{\mathfrak{s}_1}): \mathfrak{h}_1 \rightarrow \mathfrak{h}_1$ given by

$$(-i_{Z_1} \sigma_1, -D_{\mathfrak{s}_1})(r, X_1) = (-\sigma_1(Z_1, X_1), -D_{\mathfrak{s}_1} X_1), \quad \text{for } r \in \mathbb{R} \text{ and } X_1 \in \mathfrak{s}_1, \quad (37)$$

is a derivation on the Lie algebra \mathfrak{h}_1 . Denote by

$$\phi_{H_1}: \mathbb{R} \rightarrow \text{Aut}(H_1) = \text{Aut}(\mathbb{R} \odot_{(\varphi_1, \phi_1)} S_1)$$

the representation of the abelian Lie group \mathbb{R} on H_1 associated with the derivation $(-i_{Z_1} \sigma_1, D_{\mathfrak{s}_1}): \mathfrak{h}_1 \rightarrow \mathfrak{h}_1$ and by $\mathcal{M}_{H_1} \in \mathfrak{X}(H_1)$ the multiplicative vector field on H_1 whose flow is ϕ_{H_1} . Then,

$$\mathcal{M}_{H_1}(r, s_1) = \psi(r, s_1) \frac{\partial}{\partial r} \Big|_{(r, s_1)} + \mathcal{M}_{S_1}(r, s_1), \quad \text{for } (r, s_1) \in \mathbb{R} \odot_{(\varphi_1, \phi_1)} S_1 = H_1,$$

with $\psi \in \mathcal{C}^\infty(H_1)$ and $\mathcal{M}_{S_1}: H_1 \rightarrow TS_1$ a time-dependent vector field on S_1 . Since

$$0 = \overleftarrow{(Z_1, D_{\mathfrak{s}_1})(1, 0)} = [\overleftarrow{(1, 0)}, \mathcal{M}_{H_1}],$$

we conclude from (36) that

$$\mathcal{M}_{H_1}(r, s_1) = \psi(s_1) \frac{\partial}{\partial r} \Big|_{(r, s_1)} + \mathcal{M}_{S_1}(s_1),$$

with $\psi \in \mathcal{C}^\infty(S_1)$ and $\mathcal{M}_{S_1} \in \mathfrak{X}(S_1)$. Now, using (35) and the fact that \mathcal{M}_{H_1} is multiplicative, it follows that \mathcal{M}_{S_1} also is multiplicative. In addition, from (36) and (37), we deduce that the derivation on \mathfrak{s}_1 associated with \mathcal{M}_{S_1} is just $-D_{\mathfrak{s}_1}: \mathfrak{s}_1 \rightarrow \mathfrak{s}_1$. Therefore, $\mathcal{M}_{S_1} = -\mathcal{M}_1$, with \mathcal{M}_1 the multiplicative vector field on S_1 whose flow is ϕ_1 , and

$$\mathcal{M}_{H_1}(r, s_1) = \psi(s_1) \frac{\partial}{\partial r} \Big|_{(r, s_1)} - \mathcal{M}_1(s_1).$$

This implies that the flow $\phi_{H_1}: \mathbb{R} \rightarrow \text{Aut}(H_1)$ of \mathcal{M}_{H_1} has the form

$$\phi_{H_1}(t)(r, s_1) = (r + \chi_t(s_1), \phi_1(-t)(s_1))$$

and

$$\mathcal{M}_{H_1}(r, s_1) = \frac{d}{dt} \Big|_{t=0} (\chi_t(s_1)) \frac{\partial}{\partial r} \Big|_{(r, s_1)} - \mathcal{M}_1(s_1) \quad (38)$$

with $\chi \in \mathcal{C}^\infty(\mathbb{R} \times S_1)$.

In what follows, we will denote by (χ, ϕ_1) the representation ϕ_{H_1} . Since (χ, ϕ_1) is an action of \mathbb{R} on H_1 , we deduce that χ satisfies the relation

$$\chi_{t+t'}(s_1) = \chi_t(s_1) + \chi_{t'}(\phi_1(-t)s_1), \quad \text{for } t, t' \in \mathbb{R} \text{ and } s_1 \in S_1. \quad (39)$$

Moreover, using (35) and the fact that $(\chi, \phi_1)(t) \in \text{Aut}(H_1)$, for every $t \in \mathbb{R}$, we obtain that

$$\chi_t(s_1 s'_1) - \chi_t(s_1) - \chi_t(s'_1) = \frac{d}{dr} \Big|_{r=-t} \varphi_1(\phi_1(r)s_1, \phi_1(r)s'_1) - \frac{d}{dr} \Big|_{r=0} \varphi_1(\phi_1(r)s_1, \phi_1(r)s'_1). \quad (40)$$

In addition, from (36) and (38), it follows that

$$[\overleftarrow{(0, X_1)}, \mathcal{M}_{H_1}](0, \mathbf{e}_1) = \frac{d}{dt} \Big|_{t=0} (d_{\mathbf{e}_1} \chi_t)(X_1) \frac{\partial}{\partial r} \Big|_{(0, \mathbf{e}_1)} - D_{\mathbf{s}_1} X_1,$$

which, using (37), implies that

$$\frac{d}{dt} \Big|_{t=0} (d_{\mathbf{e}_1} \chi_t)(X_1) = -\sigma_1(Z_1, X_1), \quad \text{for } X_1 \in \mathfrak{s}_1. \quad (41)$$

Next, we consider the Lie group

$$S = H_1 \rtimes_{(\chi, \phi_1)} \mathbb{R} = (\mathbb{R} \odot_{(\varphi_1, \phi_1)} S_1) \rtimes_{(\chi, \phi_1)} \mathbb{R}$$

with Lie algebra

$$\mathfrak{s} = \mathfrak{h}_1 \rtimes_{(-i_{Z_1} \sigma_1, -D_{\mathbf{s}_1})} \mathbb{R} = (\mathbb{R} \odot_{D_{\mathbf{s}_1}^* \sigma_1} \mathfrak{s}_1) \rtimes_{(-i_{Z_1} \sigma_1, -D_{\mathbf{s}_1})} \mathbb{R}.$$

Then, following the construction in Section 5.3, we have that the 2-cocycle σ on \mathfrak{s} given by (5.4) is non-degenerate and it defines a left-invariant symplectic 2-form $\overleftarrow{\sigma}$ on S . Thus, S is a symplectic Lie group.

Using a similar terminology as in the Lie algebra case, we introduce the following definition.

Definition 6.10. The symplectic Lie group $(S, \overleftarrow{\sigma})$ is the *double extension* of the symplectic Lie group $(S_1, \overleftarrow{\sigma}_1)$ by the symplectic 2-cocycle $\varphi_1: S_1 \times S_1 \rightarrow \mathbb{R}$ on S_1 , the representation $\phi_1: \mathbb{R} \rightarrow \text{Aut}(S_1)$ of \mathbb{R} on S_1 and the smooth map $\chi: \mathbb{R} \times S_1 \rightarrow \mathbb{R}$, the latter satisfying (39), (40) and (41).

We remark that the discussion in this section proves the following result

Proposition 6.11. Let $(S, \overleftarrow{\sigma})$ be a connected simply connected Lie group with Lie algebra \mathfrak{s} and suppose that \mathfrak{s} is the double extension of the symplectic Lie algebra $(\mathfrak{s}_1, \sigma_1)$ by a derivation $D_{\mathbf{s}_1}: \mathfrak{s}_1 \rightarrow \mathfrak{s}_1$ and an element $Z_1 \in \mathfrak{s}_1$ satisfying (25). Then, we can choose the following objects:

- a connected, simply connected Lie group S_1 with Lie algebra \mathfrak{s}_1 ;
- a 2-cocycle $\varphi_1: S_1 \times S_1 \rightarrow \mathbb{R}$ on S_1 such that the corresponding 2-cocycle on \mathfrak{s}_1 is just σ_1 ;
- a representation $\phi_1: \mathbb{R} \rightarrow \text{Aut}(S_1)$ of \mathbb{R} on S_1 such that the corresponding derivation on \mathfrak{s}_1 is $D_{\mathbf{s}_1}$ and
- a smooth map $\chi: \mathbb{R} \times S_1 \rightarrow \mathbb{R}$ which satisfies (39), (40) and (41).

In addition, $(S, \overleftarrow{\sigma})$ is the double extension of the symplectic Lie group $(S_1, \overleftarrow{\sigma}_1)$ by φ_1 , ϕ_1 and χ .

Remark 6.12. Let $(S, \overline{\sigma})$ be a connected simply connected symplectic Lie group which is the double extension of the connected simply connected symplectic Lie group $(S_1, \overline{\sigma}_1)$ by $\varphi_1: S_1 \times S_1 \rightarrow \mathbb{R}$, $\phi_1: \mathbb{R} \rightarrow \text{Aut}(S_1)$ and $\chi: \mathbb{R} \times S_1 \rightarrow \mathbb{R}$. Suppose that Γ_{S_1} is a lattice in S_1 and p_1, q_1 are real numbers such that:

- the restriction of (φ_1, ϕ_1) to $\Gamma_{S_1} \times \Gamma_{S_1}$ takes values in the integer lattice $p_1\mathbb{Z}$ and
- the restriction to the integer lattice $q_1\mathbb{Z}$ of the representation $(\chi, \phi_1): \mathbb{R} \rightarrow \text{Aut}(\mathbb{R} \odot_{(\varphi_1, \phi_1)} S_1)$ takes values in $\text{Aut}(p_1\mathbb{Z} \odot_{(\varphi_1, \phi_1)} \Gamma_{S_1})$.

Then $\Gamma = (p_1\mathbb{Z} \odot_{(\varphi_1, \phi_1)} \Gamma_{S_1}) \rtimes_{(\chi, \phi_1)} q_1\mathbb{Z}$ is a lattice in the symplectic Lie group $S = (\mathbb{R} \odot_{(\varphi_1, \phi_1)} S_1) \rtimes_{(\chi, \phi_1)} \mathbb{R}$ and $M = \Gamma \backslash S$ is a compact manifold. Furthermore, since the symplectic structure on S is left-invariant, it induces a symplectic structure on M . Thus, M is a compact symplectic manifold.

Now, let $(S, \overline{\sigma})$ be the double extension of the symplectic Lie group $(S_1, \overline{\sigma}_1)$ by the 2-cocycle $\varphi_1: S_1 \times S_1 \rightarrow \mathbb{R}$, the representation $\phi_1: \mathbb{R} \rightarrow \text{Aut}(S_1)$ and the smooth map $\chi: \mathbb{R} \times S_1 \rightarrow \mathbb{R}$. Moreover, suppose that:

- $\varphi: S \times S \rightarrow \mathbb{R}$ is a 2-cocycle on S with values in \mathbb{R} such that the corresponding 2-cocycle on \mathfrak{s} is just σ and
- $\phi: \mathbb{R} \rightarrow \text{Aut}(S)$ is a symplectic representation of \mathbb{R} on S .

Then, one may consider the lcs extension

$$(\mathbb{R} \odot_{\varphi} S) \rtimes_{\tilde{\phi}} \mathbb{R} = \mathbb{R} \odot_{\varphi} ((\mathbb{R} \odot_{(\varphi_1, \phi_1)} S_1) \rtimes_{(\chi, \phi_1)} \mathbb{R}) \rtimes_{\tilde{\phi}} \mathbb{R}$$

of S by φ and ϕ which is a lcs Lie group of the first kind with bi-invariant Lee vector field by Example 6.6.

These results will be useful in the next section. In fact, we will see that every connected simply connected lcs nilpotent Lie group with non-zero Lee 1-form of dimension $2n$ is obtained as the lcs extension by a nilpotent representation of a connected simply connected symplectic nilpotent Lie group S of dimension $2n - 2$ and, in turn, S may be obtained by a sequence of $n - 1$ symplectic double extensions by nilpotent representations from the abelian Lie group \mathbb{R}^2 .

6.3 Lcs nilpotent Lie groups

In this section, we discuss lcs structures (with non-zero Lee 1-form) on nilpotent Lie groups.

Example 6.13. Let $(S, \overline{\sigma})$ be the double extension of the symplectic Lie group $(S_1, \overline{\sigma}_1)$ by the symplectic 2-cocycle $\varphi_1: S_1 \times S_1 \rightarrow \mathbb{R}$, the representation $\phi_1: \mathbb{R} \rightarrow \text{Aut}(S_1)$ and the smooth map $\chi: \mathbb{R} \times S_1 \rightarrow \mathbb{R}$ satisfying (39), (40) and (41).

Now, suppose that S_1 is nilpotent and that ϕ_1 also is. Then, the corresponding derivation $D_{\mathfrak{s}_1}$ on the Lie algebra \mathfrak{s}_1 of S_1 is nilpotent. Furthermore, the Lie group $H_1 = \mathbb{R} \odot_{(\varphi_1, \phi_1)} S_1$ is nilpotent and the derivation $(-i_{Z_1}\sigma_1, -D_{\mathfrak{s}_1})$ on the Lie algebra of H_1 given by (37) also is nilpotent. So, using that the Lie algebra \mathfrak{s} of S is $\mathfrak{s} = \mathfrak{h}_1 \rtimes_{(-i_{Z_1}\sigma_1, -D_{\mathfrak{s}_1})} \mathbb{R}$, we deduce that $(S, \overline{\sigma})$ is a nilpotent symplectic Lie group, the nilpotent double extension of $(S_1, \overline{\sigma}_1)$ by φ_1 , ϕ_1 and χ .

Next, let $\varphi: S \times S \rightarrow \mathbb{R}$ be a symplectic 2-cocycle on S and $\phi: \mathbb{R} \rightarrow \text{Aut}(S)$ a symplectic representation of \mathbb{R} on S . Denote by $D_{\mathfrak{s}}: \mathfrak{s} \rightarrow \mathfrak{s}$ the symplectic derivation associated with ϕ . Then, we can consider the lcs extension

$$G = (\mathbb{R} \odot_{\varphi} S) \rtimes_{\tilde{\phi}} \mathbb{R} = \mathbb{R} \odot_{\varphi} ((\mathbb{R} \odot_{(\varphi_1, \phi_1)} S_1) \rtimes_{(\chi, \phi_1)} \mathbb{R}) \rtimes_{\tilde{\phi}} \mathbb{R}$$

It is a lcs Lie group of the first kind with bi-invariant Lee vector field and with Lie algebra the lcs extension of \mathfrak{s} by σ and D ,

$$\mathfrak{g} = (\mathbb{R} \odot_{\sigma} \mathfrak{s}) \rtimes_D \mathbb{R},$$

D being the contact derivation on $\mathbb{R} \odot_{\sigma} \mathfrak{s}$ associated with the contact representation $\tilde{\phi}: \mathbb{R} \rightarrow \mathbb{R} \odot_{\phi} S$ lifting ϕ .

Assume that ϕ is nilpotent. Then, the derivation $D_{\mathfrak{s}}$ is nilpotent and, using Theorem 5.15, it follows that \mathfrak{g} is a lcs nilpotent Lie algebra with non-zero central Lee vector. Therefore, G is a lcs nilpotent Lie group with non-zero bi-invariant Lee vector field.

Next, we will see that the Lie group G in the previous example is the model of a connected simply connected lcs nilpotent Lie group with non-zero Lee 1-form. In fact, we will prove the following result.

Theorem 6.14. *Let G be a connected simply connected lcs nilpotent Lie group of dimension $2n + 2$ with non-zero Lee 1-form. Then, G is isomorphic to a lcs extension of a connected simply connected symplectic nilpotent Lie group S and, in turn, S is isomorphic to a symplectic Lie group which may be obtained as a sequence of $(n - 1)$ symplectic nilpotent double extensions from the abelian Lie group \mathbb{R}^2 .*

Proof. Let \mathfrak{g} be the Lie algebra of G . Then, using Theorem 5.15, we deduce that \mathfrak{g} is a lcs Lie algebra of the first kind with non-zero central Lee vector. Thus, G is lcs Lie group of the first kind with non-zero bi-invariant Lee vector field.

On the other hand, from Theorem 5.16, it follows that \mathfrak{g} is the lcs extension of a symplectic nilpotent Lie algebra \mathfrak{s} (of dimension $2n$) by a symplectic nilpotent derivation and, in turn, \mathfrak{s} may be obtained as a sequence of $(n - 1)$ symplectic double extensions by nilpotent derivations from the abelian Lie algebra \mathbb{R}^2 .

Denote by $\mathfrak{s}_1, \dots, \mathfrak{s}_{n-2}, \mathfrak{s}_{n-1} = \mathbb{R}^2$ the corresponding sequence of symplectic nilpotent Lie algebras and by S (respectively, S_i , with $i = 1, \dots, n - 1$) a connected simply connected nilpotent Lie group with Lie algebra \mathfrak{s} (respectively, \mathfrak{s}_i , with $i = 1, \dots, n - 1$).

Then, using Theorem 6.8 and Proposition 6.11, we deduce that G is isomorphic to a lcs extension of S and, in turn, S is isomorphic to a symplectic nilpotent Lie group which may be obtained as a sequence of $(n - 1)$ symplectic nilpotent double extensions from the abelian Lie group \mathbb{R}^2 . The corresponding sequence of $(n - 1)$ connected simply connected nilpotent symplectic Lie groups is $S_1, \dots, S_{n-2}, S_{n-1} = \mathbb{R}^2$. \square

Remark 6.15. Let S be a connected simply connected symplectic nilpotent Lie group with Lie algebra \mathfrak{s} , let $\varphi: S \times S \rightarrow \mathbb{R}$ be a symplectic 2-cocycle on S with values in \mathbb{R} and let $\phi: \mathbb{R} \rightarrow \text{Aut}(S)$ be a symplectic nilpotent representation of \mathbb{R} on S . Moreover, suppose that the structure constants of \mathfrak{s} with respect to a basis are rational numbers. It follows from Theorem 2.9 that S admits a lattice Γ_S . In addition, we will assume that we can choose two real numbers p and q such that Γ_S , p and q satisfy the conditions in Remark 6.9. Under these conditions,

$$M = ((p\mathbb{Z} \odot_{\varphi} \Gamma_S) \rtimes_{\tilde{\phi}} q\mathbb{Z}) \backslash ((\mathbb{R} \odot_{\varphi} S) \rtimes_{\tilde{\phi}} \mathbb{R})$$

is a compact lcs nilmanifold of the first kind.

Remark 6.16. A particular case of the previous situation is the following one. Suppose that S is the nilpotent double extension of a connected simply connected symplectic nilpotent Lie group S_1 with Lie algebra \mathfrak{s}_1 . Moreover, suppose that the structure constants of \mathfrak{s}_1 with respect to a basis are rational numbers. Then it follows from Theorem 2.9 that S_1 admits a lattice Γ_{S_1} .

In addition, we will assume that we can choose two real numbers p_1 and q_1 such that Γ_{S_1} , p_1 and q_1 satisfy the conditions in Remark 6.12. Under these conditions, we have that

$$\Gamma_S = (p_1 \mathbb{Z} \odot_{(\varphi_1, \phi_1)} \Gamma_{S_1}) \rtimes_{(\chi, \phi_1)} q_1 \mathbb{Z}$$

is a lattice of S . Finally, if p and q are real numbers as in Remark 6.15, we deduce that

$$M = (p \mathbb{Z} \odot_{\varphi} ((p_1 \mathbb{Z} \odot_{(\varphi_1, \phi_1)} \Gamma_{S_1}) \rtimes_{(\chi, \phi_1)} q_1 \mathbb{Z}) \rtimes_{\tilde{\phi}} q \mathbb{Z}) \backslash ((\mathbb{R} \odot_{\varphi} ((\mathbb{R} \odot_{(\varphi_1, \phi_1)} S_1) \rtimes_{(\chi, \phi_1)} \mathbb{R}) \rtimes_{\tilde{\phi}} \mathbb{R})$$

is a compact lcs nilmanifold of the first kind.

To conclude, we show how to recover the examples of Section 3 in the framework of Theorem 6.14 and Remarks 6.15 and 6.16.

We start with the 4-dimensional nilpotent Lie group G constructed in Section 3.1. In this case, S is \mathbb{R}^2 with its structure of abelian Lie group and H is the central extension $\mathbb{R} \odot_{\varphi} \mathbb{R}^2$ with respect to the 2-cocycle $\varphi: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $((x, y), (x', y')) \mapsto yx'$. Using (6), φ determines the 2-cocycle σ on \mathbb{R}^2 , which is the standard symplectic form $\sigma = dx \wedge dy$ on \mathbb{R}^2 . Further, one can see that the symplectic nilpotent representation $\phi: \mathbb{R} \rightarrow \text{Aut}(\mathbb{R}^2)$ is given by $\phi_t(x, y) = (x + ty, y)$. In the notation of Section 6.2, the function $\tilde{\chi}: \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ determining the lift of the symplectic representation ϕ_t to a contact representation $\tilde{\phi}: \mathbb{R} \rightarrow \text{Aut}(\mathbb{R} \odot_{\varphi} \mathbb{R}^2)$ is $\tilde{\chi}_t(x, y) = t \frac{y^2}{2}$, which verifies properties (32)-(34). Thus $G = (\mathbb{R} \odot_{\varphi} \mathbb{R}^2) \rtimes_{\tilde{\phi}} \mathbb{R}$ is the lcs extension of S by φ and ϕ . Concerning the lattice, we consider $\mathbb{Z} \times 2\mathbb{Z} \subset \mathbb{R}^2$; one sees that $\varphi|_{\mathbb{Z} \times 2\mathbb{Z}}$ takes values in $2\mathbb{Z}$. Then $\Xi = (2\mathbb{Z} \odot_{\varphi} (\mathbb{Z} \times 2\mathbb{Z})) \rtimes_{\tilde{\phi}} \mathbb{Z} \subset G$ is a lattice.

Let us now move to the examples of Section 3.2. In both of them, $H = \mathbb{R} \odot_{\varphi} S$ where:

- S is the 4-dimensional nilpotent Lie group with multiplication

$$(x, y, z, t) \cdot (x', y', z', t') = (x + x', y + y', z + z' + xx', t + t' + yx')$$

- $\varphi: S \times S \rightarrow \mathbb{R}$ is the 2-cocycle

$$\varphi((x, y, z, t), (x', y', z', t')) = yz' + tx'.$$

First of all, we describe S as a symplectic double extension of $S_1 = \mathbb{R}^2$. We consider \mathbb{R}^2 with global coordinates (x, z) and symplectic form $\sigma_1 = dx \wedge dz$. The data associated to the symplectic double extension of $\mathfrak{s}_1 = \mathbb{R}^2$ are, in the notation of Section 5.4, the vector $Z_1 = (0, 1)$ and the trivial derivation $D_{\mathfrak{s}_1}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Hence $\mathfrak{s} = (\mathbb{R} \oplus \mathbb{R}^2) \rtimes_{(-i_{Z_1} \sigma_1, 0)} \mathbb{R}$. As for the group structure of Section 6.2.2, we see that the 2-cocycle (φ_1, ϕ_1) is trivial, hence $S = (\mathbb{R} \times \mathbb{R}^2) \rtimes_{(\chi, \text{Id})} \mathbb{R}$, where $\chi_y(x, z) = yx$, which verifies (39), (40) and (41).

In the first example we chose the trivial symplectic representation on S and lifted it to the trivial contact representation on $H = \mathbb{R} \odot_{\varphi} S$. Hence, the nilpotent Lie group we work with is $G = (\mathbb{R} \odot_{\varphi} S) \rtimes \mathbb{R}$. Concerning the lattice, we start with $\mathbb{Z}^2 \subset \mathbb{R}^2$; then $\Gamma_S = (\mathbb{Z} \times \mathbb{Z}^2) \rtimes_{(\chi, \text{Id})} \mathbb{Z} \subset S$ is a lattice. The lattice we take in G is $(\mathbb{Z} \odot_{\varphi} \Gamma_S) \rtimes \mathbb{Z}$.

In the second example, the symplectic representation $\phi: \mathbb{R} \rightarrow \text{Aut}(S)$ is non-trivial and given by $\phi_s(x, y, z, t) = (x, y + sx, z + sy + \frac{1}{2}s^2x, t + sz + \frac{1}{2}s^2y + \frac{1}{6}s^3x)$; the function $\tilde{\chi}: \mathbb{R} \times S \rightarrow \mathbb{R}$ which determines the lift of ϕ to a contact representation $\tilde{\phi}: \mathbb{R} \rightarrow \text{Aut}(\mathbb{R} \odot_{\varphi} S)$ is

$$\tilde{\chi}_s(x, y, z, t) = s \left(xz + \frac{1}{2}y^2 - \frac{1}{3}x^3 \right) + s^2xy + \frac{1}{3}s^3x^2$$

which verifies properties (32), (33) and (34). For the lattice, we start with $\mathbb{Z} \times 6\mathbb{Z} \times \mathbb{Z} \subset \mathbb{R} \times \mathbb{R}^2$ and $\Lambda_S = (\mathbb{Z} \times 6\mathbb{Z} \times \mathbb{Z}) \rtimes_{(\chi, \text{Id})} 2\mathbb{Z} \subset S$; the restriction of φ to such subgroup takes values in $2\mathbb{Z}$. Then $(2\mathbb{Z} \odot_{\varphi} \Lambda_S) \rtimes_{\tilde{\phi}} \mathbb{Z}$ is the desired lattice.

Finally, we consider the example of Section 3.3. There one starts with the nilpotent Lie group S_{2n-2} of dimension $2n-2$ ($n \geq 4$), whose group operation is given by

$$\begin{aligned} & (x_1, y_1, \dots, x_{n-1}, y_{n-1}) \cdot (x'_1, y'_1, \dots, x'_{n-1}, y'_{n-1}) \\ &= (x_1 + x'_1, y_1 + y'_1 + x_{n-1}x'_{n-1}, x_2 + x'_2, y_2 + y'_2, \dots, x_{n-2} + x'_{n-2}, y_{n-2} + y'_{n-2}, \\ & \quad x_{n-1} + x'_{n-1}, y_{n-1} + y'_{n-1} + x_1x'_{n-1}) \end{aligned}$$

in terms of a global system of coordinates $(x_1, y_1, \dots, x_{n-1}, y_{n-1})$. The 2-cocycle $\varphi: S_{2n-2} \times S_{2n-2} \rightarrow \mathbb{R}$ is given by

$$\varphi((x_1, y_1, \dots, x_{n-1}, y_{n-1}), (x'_1, y'_1, \dots, x'_{n-1}, y'_{n-1})) = \sum_{j=1}^{n-2} x_j y'_j + y_{n-1} x'_{n-1}$$

and $H_{2n-1} = \mathbb{R} \odot_{\varphi} S_{2n-2}$.

We describe S_{2n-2} as a symplectic double extension; in the notation of Section 3.3, we consider the basis $\{\alpha_1, \dots, \alpha_{n-1}, \beta_1, \dots, \beta_{n-1}\}$ of \mathfrak{s}_{2n-2}^* with structure equations

$$d\alpha_i = 0, \quad i = 1, \dots, n-1, \quad d\beta_i = 0, \quad i = 1, \dots, n-2 \quad \text{and} \quad d\beta_{n-1} = \alpha_1 \wedge \alpha_{n-1}.$$

Hence \mathfrak{s}_{2n-2} is the direct sum of the abelian Lie algebra \mathbb{R}^{2n-6} with a 4-dimensional Lie algebra, whose dual is spanned by $\{\alpha_1, \alpha_{n-1}, \beta_1, \beta_{n-1}\}$, which is the direct sum of the 3-dimensional Heisenberg algebra and a 1-dimensional factor. We explained in the discussion of the 6-dimensional examples how this 4-dimensional Lie algebra can be described as a symplectic double extension of \mathbb{R}^2 .

In this case as well we have chosen the trivial symplectic representation on S_{2n-2} and lifted it to the trivial contact representation on H_{2n-1} . Hence $G = (\mathbb{R} \odot_{\varphi} S_{2n-2}) \times \mathbb{R}$ in this case. There is a lattice $\Xi_{2n-2} \subset S_{2n-2}$, consisting of points with integer coordinates. φ restricted to $\Xi_{2n-2} \times \Xi_{2n-2}$ takes values into \mathbb{Z} , hence $\Gamma_{2n-1} = \mathbb{Z} \odot_{\varphi} \Xi_{2n-2} \subset H_{2n-1}$ is a lattice. The lattice in G is $(\mathbb{Z} \odot_{\varphi} \Xi_{2n-2}) \times \mathbb{Z}$.

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